

QUASI-COXETER CATEGORIES AND QUANTUM GROUPS

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ABSTRACT. We define the notion of braided quasi-Coxeter category, which is informally a tensor category carrying commuting actions of a generalised braid group B_W and Artin's braid groups B_n on the tensor powers of its objects. The data which defines the action of B_W is similar to the associativity constraints in a monoidal category, but is related to the coherence of a family of fiber functors. We show that the quantum Weyl group operators of a quantised Kac-Moody algebra $U_h\mathfrak{g}$, together with the universal R -matrices of its Levi subalgebras, give rise to a braided quasi-Coxeter category structure on integrable, category \mathcal{O} -modules for $U_h\mathfrak{g}$. By relying on the 2-categorical extension of the Etingof-Kazhdan quantisation functor which we obtained in [1], we prove that this structure can be transferred to one on integrable, category \mathcal{O} -representations of \mathfrak{g} . These results will be used in the sequel to this paper to give a monodromic description of the quantum Weyl group operators of $U_h\mathfrak{g}$ which extends the one obtained by the second author for a semisimple Lie algebra.

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1. INTRODUCTION

1.1. This is the first of a series of three papers the aim of which is to extend the description of the monodromy of the rational Casimir connection in terms of quantum Weyl group operators given in [13, 14, 15] to the case of a Kac–Moody algebra \mathfrak{g} .

The method we follow is close to that of [14], and relies on the notion of braided quasi–Coxeter category, which is informally a monoidal category carrying commuting actions of a given generalized braid group and Artin’s braid groups on the tensor products of its objects. Such a structure arises for example on the category $\mathcal{O}_{U_{\hbar}\mathfrak{g}}^{\text{int}}$ of integrable, highest weight representations of a quantum Kac–Moody algebra $U_{\hbar}\mathfrak{g}$, from the quantum Weyl group operators of $U_{\hbar}\mathfrak{g}$ and the R –matrices of its Levi subalgebras.

A cohomological rigidity result, proved in the second paper of this series [2], shows that there is at most one braided quasi–Coxeter structure with prescribed local monodromies on the category $\mathcal{O}_{\mathfrak{g}}^{\text{int}}$ of integrable, highest weight representations of \mathfrak{g} .

It follows that the generalized braid group actions arising from quantum Weyl groups and the monodromy of the Casimir connection [3] are equivalent, provided the braided quasi–Coxeter structure responsible for the former can be transferred from $\mathcal{O}_{U_{\hbar}\mathfrak{g}}^{\text{int}}$ to $\mathcal{O}_{\mathfrak{g}}^{\text{int}}$. This result is one of the purposes of the present article.

1.2. The definition of a quasi–Coxeter category bears some formal similarity to that of a braided monoidal category, with Artin’s braid groups B_n replaced by a given generalised braid group B_W of Coxeter type W . If \mathcal{C} is a braided tensor category, for any object $V \in \mathcal{C}$ and $n \geq 1$, there is an action

$$\rho_b : B_n \rightarrow \text{Aut}(V_b^{\otimes n})$$

for any bracketing b on the non–associative monomial $x_1 \dots x_n$.¹ The choice of a bracketing is in a sense immaterial, since for any two bracketings b, b' , the associativity constraint $\Phi_{b'b} : V_b^{\otimes n} \rightarrow V_{b'}^{\otimes n}$ intertwines the actions of B_n .

Similarly, if V is an object in a quasi–Coxeter category \mathcal{Q} , there is an action

$$\lambda_{\mathcal{F}} : B_W \rightarrow \text{Aut}(V_{\mathcal{F}})$$

which depends on the choice of W –bracketing \mathcal{F} . Further, for any such bracketings \mathcal{F}, \mathcal{G} , there is an associativity isomorphism $\Upsilon_{\mathcal{G}\mathcal{F}} : V_{\mathcal{F}} \rightarrow V_{\mathcal{G}}$ which intertwines the B_W –actions.

1.3. The role of W –bracketings is played by the *nested sets* on the Dynkin diagram D of W [4, 14]. When $D = A_{n-1}$, the notions of bracketing and \mathfrak{S}_n –bracketing coincide. Namely, to a bracketing $x_1 \dots (x_i \dots x_j) \dots x_n$ one can associate the connected subdiagram $[i, \dots, j-1] \subset A_{n-1}$. Under this identification a (full) bracketing on $x_1 \dots x_n$ corresponds to a (maximal) nested set \mathcal{F} , *i.e.*, a (maximal) collection of *pairwise compatible*² connected subdiagrams of A_{n-1} . This definition extends to any diagram D and gives rise to the notion of W –bracketing or bracketing of type D .

¹ The notation $V_b^{\otimes n}$ indicates that n copies of V have been tensored together according to b . For example, if $b = (x_1 x_2) x_3$, one has, $V_b^{\otimes 3} = ((V \otimes V) \otimes V)$.

² For any $B, B' \in \mathcal{F}$, $B \subset B'$ or $B' \subset B$ or $B \perp B'$, *i.e.*, $B \cap B' = \emptyset$ and there is no edge connecting a vertex in B and a vertex in B' .

1.4. Despite their formal similarities, there is one significant difference between braided tensor categories and quasi-Coxeter ones. In a braided tensor category \mathcal{C} , the braid groups B_n act through morphisms of the category. In a quasi-Coxeter category \mathcal{Q} , the braid group B_W does not act by morphism in \mathcal{Q} . For example, in $\mathcal{Q} = \text{Rep } U_{\hbar}\mathfrak{g}$, the quantum Weyl group action of B_W does not commute with that of $U_{\hbar}\mathfrak{g}$. Consequently, the quantum Weyl group operators S_i^{\hbar} are not morphisms in \mathcal{Q} , but rather endomorphisms of the forgetful functor $F : \text{Rep } U_{\hbar}\mathfrak{g} \rightarrow \text{Vect}$.

This is a general feature: in a quasi-Coxeter category \mathcal{Q} the braid group B_W act by morphisms of fiber functors $F_{\mathcal{F}}$ from \mathcal{Q} to a base category \mathcal{Q}_0 , labelled by maximal nested sets \mathcal{F} on D . Specifically, for any object $V \in \mathcal{Q}$, there is a collection of morphisms

$$\lambda_{\mathcal{F}} : B_W \rightarrow \text{Aut}_{\mathcal{Q}_0}(V_{\mathcal{F}})$$

where $V_{\mathcal{F}} = F_{\mathcal{F}}(V)$. Further, for any \mathcal{F}, \mathcal{G} , there are identifications of B_W -modules $\Upsilon_{\mathcal{G}\mathcal{F}} : V_{\mathcal{F}} \rightarrow V_{\mathcal{G}}$.

1.5. Let now D be a labelled diagram. ³ A braided quasi-Coxeter category of type D consists of four pieces of data.

- (i) *Diagrammatic subcategories.* For any subdiagram $\emptyset \subseteq B \subseteq D$, a braided tensor category \mathcal{Q}_B .
- (ii) *Restriction functors.* For any $B' \subset B$, a functor of abelian categories $F_{B'B} : \mathcal{Q}_B \rightarrow \mathcal{Q}_{B'}$ such that, for any $B'' \subset B' \subset B$, $F_{B''B'} \circ F_{B'B} = F_{B''B}$. For any maximal nested set \mathcal{F} on B/B' , a tensor structure $J_{B'B}^{\mathcal{F}}$ on $F_{B'B}$, i.e., a natural isomorphism

$$J_{B'B}^{\mathcal{F}} : F_{B'B}(V_1) \otimes F_{B'B}(V_2) \rightarrow F_{B'B}(V_1 \otimes V_2)$$

For any maximal nested set \mathcal{F} on B/B' , set $F_{\mathcal{F}} = (F_{B'B}, J_{B'B}^{\mathcal{F}})$.

- (iii) *De Concini-Procesi associators.* For any maximal nested sets on \mathcal{F}, \mathcal{G} on B/B' , an isomorphism of tensor functors

$$\Upsilon_{\mathcal{G}\mathcal{F}} : F_{\mathcal{F}} \rightarrow F_{\mathcal{G}}$$

satisfying

- (a) *Horizontal factorisation.* For any $\mathcal{F}, \mathcal{G}, \mathcal{H}$ on B/B' ,

$$\Upsilon_{\mathcal{H}\mathcal{G}} \circ \Upsilon_{\mathcal{G}\mathcal{F}} = \Upsilon_{\mathcal{H}\mathcal{F}}$$

³ A labelling on a diagram D is the additional data of $m_{ij} \in \{2, \dots, \infty\}$ for any two vertices $i \neq j$ in D , such that $m_{ij} = m_{ji}$ and $m_{ij} = 2$ if $i \perp j$.

-

in $\text{Aut}(F_{\emptyset i} \otimes F_{\emptyset i})$, where c_B is the commutativity constraints in \mathcal{Q}_B .

$$B_D = \langle S_i \rangle_{i \in I} / \underbrace{S_i S_j \ S_i \cdots}_{m_{ij}} = \underbrace{S_j S_i S_j \cdots}_{m_{ij}}$$

- (i) $\lambda_{\mathcal{F}}(S_i) = S_i^Q$ if $\{i\} \in \mathcal{F}$.
- (ii) $\lambda_{\mathcal{G}} = \text{Ad}(\Upsilon_{\mathcal{GF}}) \circ \lambda_{\mathcal{F}}$.

The definition of a quasi-Coxeter category is designed so as to correspond to that of quasi-Coxeter algebras [14] under Tannakian reconstruction. Namely, for any quasi-Coxeter algebra A the category $\text{Rep } A$ should be naturally endowed with a quasi-Coxeter structure, and, conversely, a quasi-Coxeter category \mathcal{Q} should produce a quasi-Coxeter algebra structure on $\text{End}(F_{\text{op}})$.

The original definition of quasi-Coxeter algebra relies on that of D-algebra, that is, a collection of subalgebras $\{A_B\}$ indexed by connected subdiagrams $B \subset D$ such that

$$A_B \subset A_{B'} \quad \text{for any } B \subset B' \quad (1.1)$$

$$[A_B, A_{B'}] = 0 \quad \text{for any } B \perp B' \quad (1.2)$$

Property (1.1) has been easily rephrased as a collection of categories \mathcal{Q}_B , indexed by subdiagrams of D and endowed with restriction functors $\mathcal{Q}_{B'} \rightarrow \mathcal{Q}_B$ for any $B \subset B'$.

1.8. Property (1.2) is harder to translate in the language of categories. Implicitly, it asserts that the assignment $B \mapsto A_B$ extends to the case of disconnected subdiagram, according to the rule

$$A_B = \bigotimes_i A_{B_i} \quad \text{if} \quad B = \bigcup_i B_i$$

where the subdiagrams B_i 's are pairwise orthogonal. In terms of categories, this requires a generalisation of the category $\text{Rep}(A_1 \otimes A_2)$, where $A_1 \otimes A_2$ is the tensor product of two unital algebras A_1, A_2 , as a special *orthogonal* fiber products of categories, which we construct in Section 2.

1.9. The second issue concerns the existence of a D-algebra structure on any symmetrisable Kac-Moody. In the finite, affine and hyperbolic case, such structure exists, and is uniquely defined for finite and affine Kac-Moody algebras. This is, however, not the case in general, and it is easy to find counterexamples already in rank 4.

The problem amounts to finding a suitable decomposition of the Cartan subalgebra satisfying the conditions (1.1), (1.2), which is not always possible. A simple solution would be to restrict to the derived subalgebra, but to do that one would lose the non-degeneracy of the bilinear form, which is crucial in the construction of the equivalence between $O_{\mathfrak{g}}$ and $O_{U_{\mathfrak{h}}}$.

To overcome this difficulty, we give the definition of *extended* Kac-Moody algebra, along the lines of [10]. These are characterized by an enlarged Cartan subalgebra, and naturally endowed with a D-algebra structure.

1.10. **Outline of the paper.** We begin in Section 2 by defining a suitable fiber product of categories and its braided monoidal structure. In Section 3, we review a number of combinatorial notions related to diagrams. In Section 4 we lay out the axioms of D-categories and braided quasi-Coxeter categories as the Tannakian counterpart of the notion of D-algebras and quasi-Coxeter quasitriangular quasi-bialgebras. In Section 5, we define extended Kac-Moody algebras, point out that they are naturally endowed with a D-structure, and show that the category of integrable, highest weight modules over an extended quantum group has a natural structure of braided quasi-Coxeter category. Finally, in Section 6, we apply the results from [1] to the case of a Kac-Moody algebra \mathfrak{g} to obtain the desired transport of the braided quasi-Coxeter structure of the quantum group to the category of integrable category \mathcal{O} -modules for \mathfrak{g} .

2. FIBER PRODUCTS OF CATEGORIES

The following construction provides a generalisation of the category $\text{Rep}(A_1 \otimes A_2)$, where $A_1 \otimes A_2$ is the tensor product of two unital algebras A_1, A_2 , as a special fiber products of categories. In particular, we show that, under appropriate assumptions, this fiber product is defined in the 2-categories $\text{Cat}, \text{Cat}^\otimes$, *i.e.*, the 2-category of tensor categories with tensor functors and tensor natural transformations, and in DCat , *i.e.*, the 2-category of *Drinfeld categories*.

2.1. Systems of fibered categories. Let $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ be categories with functors $F_i : \mathcal{C}_i \rightarrow \mathcal{C}_0$, $i = 1, 2$. Let \mathcal{A}_{012} be the 2-category defined as follows. An object of \mathcal{A}_{012} is a tuple

$$\underline{\mathcal{C}} = (\mathcal{C}_3, F_{0,3}, F_{1,3}, F_{2,3}, \alpha_1, \alpha_2)$$

where \mathcal{C}_3 is a category, $F_{i,3} : \mathcal{C}_3 \rightarrow \mathcal{C}_i$, $i = 0, 1, 2$, are functors, and $\alpha_i : F_i \circ F_{i,3} \rightarrow F_{0,3}$ are isomorphisms such that $[\mathbf{E}_1, \mathbf{E}_2] = 0$ ⁴. In diagrams

$$\begin{array}{ccccc} & & \mathcal{C}_3 & & \\ & F_{1,3} \swarrow & \downarrow & \searrow F_{2,3} & \\ \mathcal{C}_1 & \xrightarrow{\alpha_1} & F_{0,3} & \xleftarrow{\alpha_2} & \mathcal{C}_2 \\ & F_1 \searrow & \downarrow & \swarrow F_2 & \\ & & \mathcal{C}_0 & & \end{array}$$

For every $\underline{\mathcal{C}}, \underline{\mathcal{C}}' \in \mathcal{A}_{012}$, a 1-morphism in $\mathcal{A}_{012}^{(1)}(\underline{\mathcal{C}}, \underline{\mathcal{C}}')$ is a tuple $\underline{G} = (G, \phi_0, \phi_1, \phi_2)$ with $G : \mathcal{C}_3 \rightarrow \mathcal{C}'_3$, $\phi_i : F'_{i,3} \circ G \rightarrow F_{i,3}$ such that

$$\phi_0 \circ G(\alpha'_i) = \alpha_i \circ F_i(\phi_i)$$

In diagrams

$$\begin{array}{ccc} \mathcal{C}_3 & \xrightarrow{G} & \mathcal{C}'_3 \\ \swarrow & \searrow & \swarrow \\ \mathcal{C}_1 & & \mathcal{C}_2 \\ \searrow & \swarrow & \searrow \\ & \mathcal{C}_0 & \end{array} \quad \begin{array}{ccc} \mathcal{C}_3 & \xrightarrow{G} & \mathcal{C}'_3 \\ \swarrow & \searrow & \swarrow \\ \mathcal{C}_1 & & \mathcal{C}_2 \\ \searrow & \swarrow & \searrow \\ & \mathcal{C}_0 & \end{array}$$

Finally, for every $\underline{G}, \underline{G}' \in \mathcal{A}_{012}(\underline{\mathcal{C}}, \underline{\mathcal{C}}')$, set

$$\mathcal{A}_{012}^{(2)}(\underline{G}, \underline{G}') = \{\xi : G \Rightarrow G' \mid \phi_i = \phi'_i \circ F_{i,3}(\xi)\}$$

⁴ where \mathbf{E}_i , $i = 1, 2$, are the images of $\text{End}(F_i)$ in $\text{End}(F_{0,3})$ through the maps

$$s_\bullet \mapsto \text{Ad}(\alpha_i)(s_{F_{i,3}}(\bullet))$$

2.2. A special example. We introduce the category \mathcal{C}_{012} . An objects of \mathcal{C}_{012} is a tuple $\underline{X} = (X_0, X_1, X_2, \alpha_{X,1}, \alpha_{X,2})$ where $X_i \in \mathcal{C}_i$ and $\alpha_{X,i} \in \mathcal{C}_0(F_i(X_i), X_0)^\times$, such that, for every $s_1 \in \text{End}(F_1)$ and $s_2 \in \text{End}(F_2)$, the diagram

$$\begin{array}{ccccc}
 & & F_2(X_2) & \xrightarrow{s_2, X_2} & F_2(X_2) \\
 & \nearrow \beta_{12} & \downarrow \alpha_{X,2} & & \downarrow \alpha_{X,2} \searrow \beta_{12} \\
 F_1(X_1) & \xrightarrow{\alpha_{X,1}} & X_0 & \xrightarrow{\tilde{s}_1, X_0} & X_0 \xleftarrow{\alpha_{X,1}} F_1(X_1) \\
 \downarrow s_{1, X_1} & & \downarrow \tilde{s}_2, X_0 & & \downarrow \tilde{s}_2, X_0 \\
 F_1(X_1) & \xrightarrow{\alpha_{X,1}} & X_0 & \xrightarrow{\tilde{s}_1, X_0} & X_0 \xleftarrow{\alpha_{X,1}} F_1(X_1) \\
 & \searrow \beta_{12} & \downarrow \alpha_{X,2} & & \downarrow \alpha_{X,2} \nearrow \beta_{12} \\
 & & F_2(X_2) & \xrightarrow{s_2, X_2} & F_2(X_2)
 \end{array} \tag{2.1}$$

where

$$\tilde{s}_{i, X_0} = \alpha_{X,i} \circ s_{i, X_i} \circ \alpha_{X,i}^{-1} \quad \text{and} \quad \beta_{12} = \alpha_{X,2}^{-1} \circ \alpha_{X,1}$$

is commutative.

A morphism in $\mathcal{C}_{012}(\underline{X}, \underline{Y})$ is a tuple $\underline{f} = (f_0, f_1, f_2)$ where $f_i \in \mathcal{C}_i(X_i, Y_i)$, $i = 1, 2, 3$, such that the diagram

$$\begin{array}{ccccc}
 F_1(X_1) & \xrightarrow{\alpha_{X,1}} & X_0 & \xleftarrow{\alpha_{X,2}} & F_2(X_2) \\
 F_1(f_1) \downarrow & & f_0 \downarrow & & \downarrow F_2(f_2) \\
 F_1(Y_1) & \xrightarrow{\alpha_{Y,1}} & Y_0 & \xleftarrow{\alpha_{Y,2}} & F_2(Y_2)
 \end{array}$$

is commutative, and, for every $s_1 \in \text{End}(F_1)$, $s_2 \in \text{End}(F_2)$, the diagram

$$\begin{array}{ccccc}
 X_0 & \xleftarrow{\tilde{s}_1, X_0} & X_0 & \xrightarrow{\tilde{s}_2, X_0} & X_0 \\
 f_0 \downarrow & & f_0 \downarrow & & f_0 \downarrow \\
 Y_0 & \xleftarrow{\tilde{s}_1, Y_0} & Y_0 & \xrightarrow{\tilde{s}_2, Y_0} & Y_0
 \end{array}$$

is commutative.

Lemma.

(i) *The category \mathcal{C}_{012} is naturally endowed with functors*

$$\begin{array}{ccc}
 & \mathcal{C}_{012} & \\
 F_{1,12} \swarrow & & \searrow F_{2,12} \\
 \mathcal{C}_1 & & \mathcal{C}_2 \\
 & \downarrow F_{0,12} & \\
 & \mathcal{C}_0 &
 \end{array}$$

defined by $F_{i,12}(\underline{X}) = X_i$, and $F_{i,12}(\underline{f}) = f_i$.

(ii) *There are natural isomorphisms of functors*

$$\alpha_{\bullet,1} : F_1 \circ F_{1,12} \rightarrow F_{0,12} \quad \text{and} \quad \alpha_{\bullet,2} : F_2 \circ F_{2,12} \rightarrow F_{0,12}$$

$$\begin{array}{ccccc}
 & & \mathcal{C}_{012} & & \\
 & \swarrow F_{1,12} & \downarrow & \searrow F_{2,12} & \\
 \mathcal{C}_1 & \xrightarrow{\quad} & F_{0,12} & \xleftarrow{\quad} & \mathcal{C}_2 \\
 & \searrow F_1 & \downarrow & \swarrow F_2 & \\
 & & \mathcal{C}_0 & &
 \end{array}$$

(iii) *Let E_i , $i = 1, 2$, be the images of $\text{End}(F_i)$ in $\text{End}(F_{0,12})$ through $\alpha_{\bullet,i}$. Then*

$$[E_1, E_2] = 0$$

Proof. It follows directly from the definition. \square

Corollary. $\underline{\mathcal{C}}_{012} = (\mathcal{C}_{012}, F_{0,12}, F_{1,12}, F_{2,12}, \alpha_{\bullet,1}, \alpha_{\bullet,2}) \in \mathcal{A}_{012}$.

2.3. Orthogonal fibered product of categories.

Definition. Let $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$ be k -linear additive categories, and let $F_i : \mathcal{C}_i \rightarrow \mathcal{C}_0$, $i = 1, 2$ be two functors. The *orthogonal 2-fiber product of \mathcal{C}_1 and \mathcal{C}_2 over \mathcal{C}_0* is the object

$$\text{OFP}(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2) \in \mathcal{A}_{012}$$

satisfying the following universal property. For every $\underline{\mathcal{C}} \in \mathcal{A}_{012}$, there exists a unique (up to a unique isomorphism) $\underline{F} \in \mathcal{A}_{012}(\underline{\mathcal{C}}, \text{OFP}(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2))$.

Remark. $\text{OFP}(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2)$ is defined up to an equivalence, which is uniquely defined, up to a unique isomorphism.

Proposition.

$$\text{OFP}(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2) = \mathcal{C}_{012}$$

Proof. We have to show that for every $\underline{\mathcal{C}} = (\mathcal{C}_3, F_{0,3}, F_{1,3}, F_{2,3}, \gamma_1, \gamma_2) \in \mathcal{A}_{012}$ there exist a unique (up to isomorphism) $\underline{F} = (F, \phi_0, \phi_1, \phi_2) \in \mathcal{A}_{012}(\underline{\mathcal{C}}, \underline{\mathcal{C}}_{012})$.

There is a natural functor $F : \mathcal{C}_3 \rightarrow \mathcal{C}_{012}$ defined for every $X \in \mathcal{C}_3$ by

$$F(X) = (F_{0,3}(X), F_{1,3}(X), F_{2,3}(X), \gamma_{1,X}, \gamma_{2,X})$$

and for every $f \in \mathcal{C}_3(X, Y)$ by

$$F(f) = (F_{0,3}(f), F_{1,3}(f), F_{2,3}(f))$$

Since $F_{i,12} \circ F_{i,3}(X) = F_{i,3}(X)$, for all $X \in \mathcal{C}_3$, one chooses $\phi_i : F_{i,12} \circ F \rightarrow F_{i,3}$ to be the identity, and we set

$$\underline{F} = (F, \text{id}, \text{id}, \text{id}) \in \mathcal{A}_{012}(\underline{\mathcal{C}}, \underline{\mathcal{C}}_{012})$$

Assume now there is another morphism $(G, \phi_0, \phi_1, \phi_2) \in \mathcal{A}_{012}(\underline{\mathcal{C}}, \underline{\mathcal{C}}_{012})$, where

$$G(X) = (G(X)_0, G(X)_1, G(X)_2, \alpha_{1,X}^G, \alpha_{2,X}^G)$$

for every $X \in \mathcal{C}_3$. By definition, $\phi_i : F_{i,12} \circ G \rightarrow F_{i,3}$, $i = 0, 1, 2$, are isomorphisms such that

$$\phi_0 \circ \alpha_{i,\bullet}^G = \gamma_i \circ F_i(\phi_i)$$

Since $F_{i,12} \circ G(X) = G(X)_i$, $i = 0, 1, 2$, it follows that the triple (ϕ_0, ϕ_1, ϕ_2) satisfies

$$\begin{array}{ccccc} F_1(G(X)_1) & \xrightarrow{\alpha_{1,X}^G} & G(X)_0 & \xleftarrow{\alpha_{2,X}^G} & F_2(G(X)_2) \\ F_1(\phi_1) \downarrow & & \phi_0 \downarrow & & \downarrow F_2(\phi_2) \\ F_1 \circ F_{1,3}(X) & \xrightarrow{\gamma_{1,X}} & F_{0,3}(X) & \xleftarrow{\gamma_{2,X}} & F_2 \circ F_{2,3}(X) \end{array}$$

and it defines an isomorphism in \mathcal{C}_{012}

$$\xi_X = (\phi_0, \phi_1, \phi_2)_X : G(X) \rightarrow F(X)$$

Finally, $\xi : (G, \phi_0, \phi_1, \phi_2) \rightarrow (F, \text{id}, \text{id}, \text{id})$ is an isomorphism in $\mathcal{A}_{012}^{(2)}$. \square

Corollary. *Let A_1, A_2 be two unital \mathbf{k} -algebras, and set $\mathcal{C}_0 = \text{Vect}_{\mathbf{k}}$*

$$\mathcal{C}_i = \text{Rep}(A_i) \quad \text{and} \quad F_i(V, \pi_V) = V$$

Then there is an equivalence of \mathbf{k} -linear categories

$$\mathcal{C}_{012} \simeq \text{Rep}(A_1 \otimes A_2)$$

Proof. It is immediate to show that the category $\text{Rep}(A_1 \otimes A_2)$ satisfies the same universal property of $\text{OFP}(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2)$. The equivalence follows.

More specifically, let $\underline{V} = (V_0, V_1, V_2, \alpha_{V,1}, \alpha_{V,2})$ be an object of \mathcal{C}_{012} . We can use the isomorphisms of vector spaces $\alpha_{V,i}$ to induce an action of A_i on V_0 , so that now $\alpha_{V,i}$ are morphism of representations of A_i from $V_i \rightarrow V_0$. The tuple

$$(\text{id}, \alpha_{V,1}, \alpha_{V,2}) : (V_0, V_1, V_2, \alpha_{V,1}, \alpha_{V,2}) \rightarrow (V_0, V_0, V_0, \text{id}, \text{id})$$

is then an isomorphism in \mathcal{C}_{012} . Moreover, we observe that all morphisms in \mathcal{C}_{012} between $(V_0, V_0, V_0, \text{id}, \text{id})$ and $(W_0, W_0, W_0, \text{id}, \text{id})$ are of the form (f_0, f_0, f_0) where $f_0 : V_0 \rightarrow W_0$.

There is a natural functor

$$\text{Rep}(A_1 \otimes A_2) \rightarrow \mathcal{C}_{012}$$

mapping

$$V \mapsto (V, \text{res}_1 V, \text{res}_2 V, \text{id}, \text{id})$$

By the previous observation, this functor is fully faithful and essentially surjective. \square

Remark. The construction of the *orthogonal* 2-fiber product can be similarly performed without the condition $[\mathbf{E}_1, \mathbf{E}_2] = 0$. This gives the 2-fiber product in \mathbf{Cat} , denoted $\tilde{\mathcal{C}}_{012} = \text{FP}(\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2)$, and it generalises the category $\text{Rep}(A_1 \star A_2)$, where $A_1 \star A_2$ is the free product of two unital algebras A_1, A_2 .

2.4. Tensor structures on 2-fiber products. Assume $(\mathcal{C}_i, \otimes, \Phi_i)$, $i = 0, 1, 2$, are tensor categories and $F_i : \mathcal{C}_i \rightarrow \mathcal{C}_0$, $i = 1, 2$, are tensor functors with tensor structures $J_i : \otimes \circ F_i \times F_i \rightarrow F_i \circ \otimes$.⁵

⁵For $i = 1, 2$, it holds

$$J_{i,1,23} \circ J_{i,2,3} \circ \Phi_0 = F_i(\Phi_i) \circ J_{i,12,3} \circ J_{i,1,2}$$

Proposition. *The category $\tilde{\mathcal{C}}_{012}$ is naturally a tensor category with a tensor structure on the functor $F_{0,12}$ compatible with the morphisms α_1, α_2 .*

Proof. The tensor product on $\tilde{\mathcal{C}}_{012}$ is described as follows. Let $\underline{X} = (X_0, X_1, X_2, \alpha_{1,X}, \alpha_{2,X})$ and $\underline{Y} = (Y_0, Y_1, Y_2, \alpha_{1,Y}, \alpha_{2,Y})$ be two objects in \mathcal{C}_{012} . Then

$$\underline{X} \otimes \underline{Y} = (X_0 \otimes Y_0, X_1 \otimes Y_1, X_2 \otimes Y_2, \beta_{1,X,Y}, \beta_{2,X,Y})$$

where $\beta_{i,X,Y} : F_i(X_i \otimes Y_i) \rightarrow X_0 \otimes Y_0$, $i = 1, 2$,

$$\beta_{i,X,Y} = (\alpha_{i,X} \otimes \alpha_{i,Y}) \circ J_{i,X,Y}^{-1}$$

The associativity constraint in $\tilde{\mathcal{C}}_{012}$ is obtained by those of $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2$. Namely, the associativity constraint in $\tilde{\mathcal{C}}_{012}$ is the natural isomorphism

$$\Phi_{012} = (\Phi_0, \Phi_1, \Phi_2)$$

which satisfies

$$\begin{array}{ccccc} F_1((X_1 \otimes Y_1) \otimes Z_1) & \xrightarrow{\gamma_1^{12,3}} & (X_0 \otimes Y_0) \otimes Z_0 & \xleftarrow{\gamma_2^{12,3}} & F_2((X_2 \otimes Y_2) \otimes Z_2) \\ \downarrow F_1(\Phi_1) & & \downarrow \Phi_0 & & \downarrow F_2(\Phi_2) \\ F_1(X_1 \otimes (Y_1 \otimes Z_1)) & \xrightarrow{\gamma_1^{1,23}} & X_0 \otimes (Y_0 \otimes Z_0) & \xleftarrow{\gamma_1^{1,23}} & F_2(X_2 \otimes (Y_2 \otimes Z_2)) \end{array}$$

where, by construction,

$$\gamma_i^{12,3} = ((\alpha_{i,X} \otimes \alpha_{i,Y}) \otimes \alpha_{i,Z}) \circ J_{i,X,Y}^{-1} \circ J_{i,X \otimes Y,Z}^{-1}$$

and

$$\gamma_i^{1,23} = (\alpha_{i,X} \otimes (\alpha_{i,Y} \otimes \alpha_{i,Z})) \circ J_{i,Y,Z}^{-1} \circ J_{i,X,Y \otimes Z}^{-1}$$

□

Remark. The construction above is a generalisation of the following situation. Let (A_i, Φ_i, J_i) , $i = 1, 2$, be two quasibialgebras with associators Φ_i and twists J_i . Denote by A_i^J the bialgebra obtained from A_i by twisting with J_i . Set $\mathcal{C}_i = \text{Rep}(A_i)$ and $\mathcal{C}_0 = \text{Vect}$. The twist J_i defines a tensor structure on the forgetful functor $\mathcal{C}_i \rightarrow \mathcal{C}_0$. It is easy to see that, as tensor categories fibered over \mathcal{C}_0 ,

$$\text{FP}(\text{Vect}, \text{Rep}(A_1), \text{Rep}(A_2)) = \text{Rep}(A_1^J \star A_2^J)$$

2.5. Tensor structure on orthogonal 2-fiber products. The tensor product on $\tilde{\mathcal{C}}_{012}$ does not automatically restrict to \mathcal{C}_{012} , because the condition (2.1) does not extend, in full generality, to tensor products in $\tilde{\mathcal{C}}_{012}$. To overcome this obstruction, we consider the following situation, inspired by the case of a category enriched over the category of topological vector spaces with a topologically complete tensor product.

Let (\mathcal{H}, \otimes) be a complete tensor category. A tensor \mathcal{H} -category (\mathcal{C}_0, \otimes) is \mathcal{H} -complete if, for every $X_i, Y_i \in \mathcal{C}_0$,

$$\mathcal{C}_0(X_1 \otimes X_2, Y_1 \otimes Y_2) \simeq \mathcal{C}_0(X_1, Y_1) \otimes \mathcal{C}_0(X_2, Y_2)$$

Under these assumptions, for every functor $F : \mathcal{C} \rightarrow \mathcal{C}_0$, one has

$$\text{End}(F \times F) \simeq \text{End}(F) \otimes \text{End}(F)$$

From now on, let \mathcal{C}_0 be a fixed \mathcal{H} -complete category. Assume $(\mathcal{C}_i, \otimes, \Phi_i)$, $i = 1, 2$, are tensor categories and $F_i : \mathcal{C}_i \rightarrow \mathcal{C}_0$, $i = 1, 2$, are tensor functors with tensor structures $J_i : \otimes \circ F_i \times F_i \rightarrow F_i \circ \otimes$.

Proposition. *The category \mathcal{C}_{012} is naturally a tensor category with a tensor structure on the functor $F_{0,12}$ compatible with the morphisms α_1, α_2 .*

Proof. Since the base category is \mathcal{H} -complete, the condition (2.1) is automatically satisfied on tensor products, and Proposition 2.4 naturally extends to \mathcal{C}_{012} . \square

Remark. The construction above is a generalisation of the following situation. Let (A_i, Φ_i, J_i) , $i = 1, 2$, be two quasibialgebras with associators Φ_i and twists J_i . Denote by A_i^J the bialgebra obtained from A_i by twisting with J_i . It is easy to see that, as tensor categories fibered over \mathcal{C}_0 ,

$$\mathrm{FP}(\mathrm{Vect}, \mathrm{Rep}(A_1), \mathrm{Rep}(A_2)) \simeq \mathrm{Rep}(A_1^J \otimes A_2^J)$$

Next, we generalise the construction of the tensor category

$$(\mathrm{Rep}(A_1 \otimes A_2), \otimes, \Phi_1 \Phi_2)$$

2.6. Orthogonal fiber product of Drinfeld categories. Assume $(\mathcal{C}_0, \otimes, \Phi_0)$ is a \mathcal{H} -complete tensor category. A Drinfeld category over \mathcal{C}_0 is a tensor category $(\mathcal{C}, \otimes, \Phi^0)$ fibered over \mathcal{C}_0 , endowed with an additional associativity constraint $\Phi \neq \Phi^0$. Further, we require the functor

$$(F, J^0) : (\mathcal{C}, \otimes, \Phi^0) \rightarrow (\mathcal{C}_0, \otimes, \Phi_0)$$

to be **End**-full.⁶ A Drinfeld category is fibered over \mathcal{C}_0 if it is endowed with an additional tensor structure J on F with respect to the associativity constraint Φ .

Proposition. *The orthogonal fiber product of two (fibered) Drinfeld categories over \mathcal{C}_0 is a (fibered) Drinfeld category.*

Proof. Let $(\mathcal{C}_i, \otimes, \Phi_i^{(0)}, \Phi_i)$ be Drinfeld categories, with tensor functors

$$(F_i, J_i^0) : (\mathcal{C}_i, \otimes, \Phi_i^{(0)}) \rightarrow (\mathcal{C}_0, \otimes, \Phi_0)$$

for $i = 1, 2$. The category \mathcal{C}_{012} is endowed with a *basic* tensor structure, and is fibered over \mathcal{C}_0 , as explained in the previous paragraph. Therefore it remains to show that the non trivial associativity constraints Φ_1, Φ_2 induce a non trivial associativity constraint on \mathcal{C}_{012} .

Let $F^n : \mathcal{C} \times \cdots \times \mathcal{C} \rightarrow \mathcal{C}_0$ be the functor

$$(X_1, \dots, X_n) \mapsto (\cdots (F(X_1) \otimes F(X_2)) \otimes \cdots \otimes F(X_{n-1})) \otimes F(X_n)$$

For any bracketing $b \in \mathcal{B}_n$, $\phi_{0,b} : F^n \rightarrow F_b^n$ is the unique isomorphism determined by the associativity constraint of \mathcal{C}_0 . This provides a canonical identification of the monoids $\mathrm{End}(F_b^n)$ for any $b \in \mathcal{B}_n$.

Lemma. *There are morphisms of monoids*

$$\mathrm{End}(F_1^n) \rightarrow \mathrm{End}(F_{2,12}^n) \quad \text{and} \quad \mathrm{End}(F_2^n) \rightarrow \mathrm{End}(F_{1,12}^n)$$

⁶A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to be **End**-full if the natural lift $\tilde{F} : \mathcal{C} \rightarrow \mathrm{Rep}_{\mathcal{D}} \mathrm{End}(F)$ is full, where $\mathrm{Rep}_{\mathcal{D}} \mathrm{End}(F)$ denotes the category of *modules* over the monoid $\mathrm{End}(F)$ in the category \mathcal{D} .

Proof. The case $n = 1$ follows directly from the commutativity of $\text{End}(F_1), \text{End}(F_2)$ in $\text{End}(F_{0,12})$, and from the End -fullness of F_1, F_2 . By \mathcal{H} -completeness, the result extends to $n > 1$. \square

Let $\tilde{\Phi}_i \in \text{End}(F_{0,12}^3)$ be the endomorphism induced by $\Phi_i^0 \circ \Phi_i$ through J_i^0 . One has

$$[\tilde{\Phi}_1, \tilde{\Phi}_2] = 0$$

in $\text{End}(F_{0,12}^3)$, and by End -fullness there exist morphisms in \mathcal{C}_1 and \mathcal{C}_2 corresponding to $\tilde{\Phi}_2$ and $\tilde{\Phi}_1$, respectively. It follows that

$$\Phi_0^{-1} \circ \tilde{\Phi}_1 \circ \tilde{\Phi}_2 = \Phi_0^{-1} \circ \tilde{\Phi}_2 \circ \tilde{\Phi}_1$$

defines a morphism in \mathcal{C}_{012} , which satisfies the pentagon axiom and defines a non-trivial associativity constraint on \mathcal{C}_{012} .

Let now $\mathcal{C}_1, \mathcal{C}_2$ be Drinfeld categories fibered over \mathcal{C}_0 with tensor structures J_1, J_2 , respectively. The functor $F_{0,12}$ is then endowed with a natural tensor structure $\tilde{J}_1 \circ \tilde{J}_2 = \tilde{J}_2 \circ \tilde{J}_1$ with respect to the associativity constraint $\tilde{\Phi}_1 \circ \tilde{\Phi}_2$. \square

3. DIAGRAMS AND NESTED SETS

We review in this section a number of combinatorial notions associated to a diagram D , in particular the definition of nested sets on D and of the De Concini–Procesi associahedron of D following [4] and [14, Section 2].

3.1. Nested sets on diagrams. By a *diagram* we shall mean a nonempty undirected graph D with no multiple edges or loops. We denote the set of vertices of D by $\mathbf{V}(D)$ and set $|D| = |\mathbf{V}(D)|$. A *subdiagram* $B \subseteq D$ is a full subgraph of D , that is, a graph consisting of a subset $\mathbf{V}(B)$ of vertices of D , together with all edges of D joining any two elements of $\mathbf{V}(B)$. We will often abusively identify such a B with its set of vertices and write $i \in B$ to mean $i \in \mathbf{V}(B)$.

The union $B_1 \cup B_2$ of two subdiagrams $B_1, B_2 \subseteq D$ is the subdiagram having $\mathbf{V}(B_1) \cup \mathbf{V}(B_2)$ as its set of vertices. Two subdiagrams $B_1, B_2 \subseteq D$ are *orthogonal* if $\mathbf{V}(B_1) \cap \mathbf{V}(B_2) = \emptyset$ and no two vertices $i \in B_1, j \in B_2$ are joined by an edge in D . B_1 and B_2 are *compatible* if either one contains the other or they are orthogonal.

Definition. A *nested set* on a diagram D is a collection \mathcal{H} of pairwise compatible, connected subdiagrams of D which contains the connected components D_1, \dots, D_r of D .

3.2. The De Concini–Procesi associahedron. Let \mathcal{N}_D be the partially ordered set of nested sets on D , ordered by reverse inclusion. \mathcal{N}_D has a unique maximal element $\mathbf{1} = \{D_i\}$ and its minimal elements are the *maximal nested sets*. We denote the set of maximal nested sets on D by $\text{Mns}(D)$. Every nested set \mathcal{H} on D is uniquely determined by a collection $\{\mathcal{H}_i\}_{i=1}^r$ of nested sets on the connected components of D . We therefore obtain canonical identifications

$$\mathcal{N}_D = \prod_{i=1}^r \mathcal{N}_{D_i} \quad \text{and} \quad \text{Mns}(D) = \prod_{i=1}^r \text{Mns}(D_i)$$

The *De Concini–Procesi associahedron* \mathcal{A}_D is the regular CW-complex whose poset of (nonempty) faces is \mathcal{N}_D . It easily follows from the definition that

$$\mathcal{A}_D = \prod_{i=1}^r \mathcal{A}_{D_i}$$

It can be realized as a convex polytope of dimension $|D| - r$. For any $\mathcal{H} \in \mathcal{N}_D$, we denote by $\dim(\mathcal{H})$ the dimension of the corresponding face in \mathcal{A}_D .

3.3. The rank function of \mathcal{N}_D . For any nested set \mathcal{H} on D and $B \in \mathcal{H}$, we set $i_{\mathcal{H}}(B) = \bigcup_{i=1}^m B_i$ where the B_i 's are the maximal elements of \mathcal{H} properly contained in B .

Definition. Set $\underline{\alpha}_{\mathcal{H}}^B = B \setminus i_{\mathcal{H}}(B)$. We denote by

$$n(B; \mathcal{H}) = |\underline{\alpha}_{\mathcal{H}}^B| \quad \text{and} \quad n(\mathcal{H}) = \sum_{B \in \mathcal{H}} (n(B; \mathcal{H}) - 1)$$

An element $B \in \mathcal{H}$ is called *unsaturated* if $n(B; \mathcal{H}) > 1$.

Proposition.

(i) For any nested set $\mathcal{H} \in \mathcal{N}_D$,

$$n(\mathcal{H}) = |D| - |\mathcal{H}| = \dim(\mathcal{H})$$

(ii) If \mathcal{H} is a maximal nested set if and only if $n(B; \mathcal{H}) = 1$ for any $B \in \mathcal{H}$.

(iii) Any maximal nested set is of cardinality $|D|$.

For any $\mathcal{F} \in \text{Mns}(D)$, $B \in \mathcal{F}$, $i_{\mathcal{F}}(B)$ denotes the maximal element in \mathcal{F} properly contained in B and $\underline{\alpha}_{\mathcal{F}}^B = B \setminus i_{\mathcal{F}}(B)$ consists of one vertex, denoted $\alpha_{\mathcal{F}}^B$.

For any $\mathcal{F} \in \text{Mns}(D)$, $B \in \mathcal{F}$, we denote by $\mathcal{F}_B \in \text{Mns}(B)$ the maximal nested set induced by \mathcal{F} on B .

3.4. Quotient diagrams. Let $B \subsetneq D$ a proper subdiagram with connected components B_1, \dots, B_m .

Definition. The set of vertices of the quotient diagram D/B is $V(D) \setminus V(B)$. Two vertices $i \neq j$ of D/B are linked by an edge if and only if the following holds in D

$$i \not\leq j \quad \text{or} \quad i, j \not\leq B_i \quad \text{for some } i = 1, \dots, m$$

For any connected subdiagram $C \subseteq D$ not contained in B , we denote by $\overline{C} \subseteq D/B$ the connected subdiagram with vertex set $V(C) \setminus V(B)$.

3.5. Compatible subdiagrams of D/B .

Lemma. Let $C_1, C_2 \not\subseteq B$ be two connected subdiagrams of D which are compatible. Then

- (i) $\overline{C}_1, \overline{C}_2$ are compatible unless $C_1 \perp C_2$ and $C_1, C_2 \not\leq B_i$ for some i .
- (ii) If C_1 is compatible with every B_i , then \overline{C}_1 and \overline{C}_2 are compatible.

In particular, if \mathcal{F} is a nested set on D containing each B_i , then $\overline{\mathcal{F}} = \{\overline{C}\}$, where C runs over the elements of \mathcal{F} such that $C \not\subseteq B$, is a nested set on D/B .

Let now A be a connected subdiagram of D/B and denote by $\tilde{A} \subseteq D$ the connected subdiagram with vertex set

$$V(\tilde{A}) = V(A) \cup \bigcup_{i: B_i \not\subseteq V(A)} V(B_i)$$

Clearly, $A_1 \subseteq A_2$ or $A_1 \perp A_2$ imply $\tilde{A}_1 \subseteq \tilde{A}_2$ and $\tilde{A}_1 \perp \tilde{A}_2$ respectively, so the lifting map $A \rightarrow \tilde{A}$ preserves compatibility.

3.6. Nested sets on quotients. For any connected subdiagrams $A \subseteq D/B$ and $C \subseteq D$, we have

$$\overline{\tilde{A}} = A \quad \text{and} \quad \tilde{\overline{C}} = C \cup \bigcup_{i: B_i \not\subseteq C} B_i$$

In particular, $\tilde{\overline{C}} = C$ if, and only if, C is compatible with B_1, \dots, B_m and not contained in B . The applications $C \rightarrow \overline{C}$ and $A \rightarrow \tilde{A}$ therefore yield a bijection between the connected subdiagrams of D which are either orthogonal to or strictly contain each B_i and the connected subdiagrams of D/B . This bijection preserves compatibility and therefore induces an embedding $\mathcal{N}_{D/B} \hookrightarrow \mathcal{N}_D$. This yields an embedding

$$\mathcal{N}_{D/B} \times \mathcal{N}_B = \mathcal{N}_{D/B} \times (\mathcal{N}_{B_1} \times \dots \times \mathcal{N}_{B_m}) \hookrightarrow \mathcal{N}_D$$

with image the poset of nested sets on D containing each B_i . Similarly, for any $B \subseteq B' \subseteq B''$, we obtain a map

$$\cup : \mathcal{N}_{B''/B'} \times \mathcal{N}_{B'/B} \hookrightarrow \mathcal{N}_{B''/B}$$

The map \cup restricts to maximal nested sets. For any $B \subset B'$, we denote by $\text{Mns}(B', B)$ the collection of maximal nested sets on B'/B . Therefore, for any $B \subset B' \subset B''$, we obtain an embedding

$$\cup : \text{Mns}(B'', B') \times \text{Mns}(B', B) \rightarrow \text{Mns}(B'', B)$$

such that, for any $\mathcal{F} \in \text{Mns}(B'', B')$, $\mathcal{G} \in \text{Mns}(B', B)$,

$$(\mathcal{F} \cup \mathcal{G})_{B'/B} = \mathcal{G}$$

3.7. Elementary and equivalent pairs.

Definition. An ordered pair $(\mathcal{G}, \mathcal{F})$ in $\text{Mns}(D)$ is called *elementary* if \mathcal{G} and \mathcal{F} differ by one element. A sequence $\mathcal{H}_1, \dots, \mathcal{H}_m$ in $\text{Mns}(D)$ is called *elementary* if $|\mathcal{H}_{i+1} \setminus \mathcal{H}_i| = 1$ for any $i = 1, 2, \dots, m-1$.

Definition. The *support* $\text{supp}(\mathcal{F}, \mathcal{G})$ of an elementary pair in $\text{Mns}(D)$ is the unique unsaturated element of $\mathcal{F} \cap \mathcal{G}$. The *central support* $\mathfrak{z}\text{supp}(\mathcal{F}, \mathcal{G})$ is the union of the maximal elements of $\mathcal{F} \cap \mathcal{G}$ properly contained in $\text{supp}(\mathcal{F}, \mathcal{G})$. Thus

$$\mathfrak{z}\text{supp}(\mathcal{F}, \mathcal{G}) = \text{supp}(\mathcal{F}, \mathcal{G}) \setminus \alpha_{\mathcal{F} \cap \mathcal{G}}^{\text{supp}(\mathcal{F}, \mathcal{G})}$$

Definition. Two elementary pairs $(\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')$ in $\text{Mns}(D)$ are *equivalent* if

$$\begin{aligned} \text{supp}(\mathcal{F}, \mathcal{G}) &= \text{supp}(\mathcal{F}', \mathcal{G}') \\ \alpha_{\mathcal{F}}^{\text{supp}(\mathcal{F}, \mathcal{G})} &= \alpha_{\mathcal{F}'}^{\text{supp}(\mathcal{F}', \mathcal{G}')} & \alpha_{\mathcal{G}}^{\text{supp}(\mathcal{F}, \mathcal{G})} &= \alpha_{\mathcal{G}'}^{\text{supp}(\mathcal{F}', \mathcal{G}')} \end{aligned}$$

3.8. Nested sets and chains of subdiagrams.

Definition. A *chain* from $B' \subseteq D$ to $B \subset B'$ is a sequence of subdiagrams

$$C : B = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_m = B'$$

A chain is called *maximal* if $|B_k \setminus B_{k-1}| = 1$ for every k . The sets of chains and maximal chains from B' to B are denoted $\text{Ch}(B', B)$ and $\text{MCh}(B', B)$, respectively.

Lemma. *There is a surjection $\iota : \text{Ch}(B', B) \rightarrow \text{Ns}(B', B)$ given by*

$$\iota(B = B_0 \subsetneq B_1 \subsetneq \cdots \subsetneq B_m = B') = \bigcup_{k=1}^m \text{conn}(B_k)$$

where $\text{conn}(B_k)$ denotes the connected components of B_k . The restriction of ι to maximal chains gives a surjection $\iota : \text{MCh}(B', B) \rightarrow \text{Mns}(B', B)$

4. QUASI-COXETER CATEGORIES

The goal of this section is to rephrase the notion of quasi-Coxeter quasitriangular quasibialgebra defined in [14] in terms of categories of representations.

4.1. D -categories. Let D be a diagram. A D -algebra is a pair $(A, \{A_B\}_{B \in \text{CSD}(D)})$, where A is an associative algebra and $\{A_B\}_{B \in \text{CSD}(D)}$ is a collection of subalgebras indexed by $\text{CSD}(D)$ and satisfying

$$A_B \subseteq A_{B'} \quad \text{if } B \subseteq B' \quad \text{and} \quad [A_B, A_{B'}] = 0 \quad \text{if } B \perp B'$$

The following rephrases the notion of D -algebras in terms of their category of representations.

Definition. A D -category

$$\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\})$$

is the datum of

- a collection of k -linear additive categories $\{\mathcal{C}_B\}_{B \subseteq D}$;
- for any pair of subdiagrams $B \subseteq B'$, an additive k -linear functor $F_{BB'} : \mathcal{C}_{B'} \rightarrow \mathcal{C}_B$;

satisfying the following properties

- for any $B \subseteq D$, $F_{BB} = \text{id}_{\mathcal{C}_B}$;
- for any $B \subseteq B' \subseteq B''$, $F_{BB'} \circ F_{B'B''} = F_{BB''}$;
- for any $B' \perp B''$,

$$\mathcal{C}_{B' \cup B''} = \text{OPF}(\mathcal{C}_\emptyset, \mathcal{C}_{B'}, \mathcal{C}_{B''})$$

4.2. Some remarks about Definition 4.1.

4.2.1. It may seem more natural to replace the equality of functors $F_{BB'} \circ F_{B'B''} = F_{BB''}$ by the existence of invertible natural transformations $\alpha_{BB''}^{B'} : F_{BB'} \circ F_{B'B''} \Rightarrow F_{BB''}$ for any $B \subseteq B'$ satisfying the associativity constraints $\alpha_{BB'''}^{B'} \circ F_{BB'}(\alpha_{B'B'''}^{B''}) = \alpha_{BB'''}^{B''} \circ (\alpha_{BB''}^{B'})_{F_{B''B'''}}$ for any $B \subseteq B' \subseteq B'' \subseteq B'''$. A simple coherence argument shows however that this leads to a notion of D -category which is equivalent to the one given above.

4.2.2. We will usually think of \mathcal{C}_\emptyset as a base category and at the functors F as forgetful functors. Then the family of algebras E_B defines a structure of D -algebra on E_D . Conversely, every D -algebra A admits such a description setting $\mathcal{C}_B = \text{Rep } A_B$ for $B \neq \emptyset$ and $\mathcal{C}_\emptyset = \text{Vect}_k$, $F_{BB'} = i_{B'B}^*$, where $i_{B'B} : A_B \subset A_{B'}$ is the inclusion.

4.2.3. The above definition of D -category may be rephrased as follows. Let $I(D)$ be the category whose objects are subdiagrams $B \subseteq D$ and morphisms $B' \rightarrow B$ the inclusions $B \subset B'$. Then a D -category is a functor

$$\mathcal{C} : I(D) \rightarrow \text{Cat}$$

preserving orthogonality, i.e., satisfying

$$\mathcal{C}_{B' \cup B''} = \text{OFP}(\mathcal{C}_\emptyset, \mathcal{C}_{B'}, \mathcal{C}_{B''})$$

for every $B' \perp B''$.

4.3. **Strict morphisms of D -categories.** The interpretation of D -categories in terms of $I(D)$ suggests that a morphism of D -categories $\mathcal{C}, \mathcal{C}'$ is one of the corresponding functors

$$\begin{array}{ccc} & \mathcal{C} & \\ \curvearrowright & \downarrow & \curvearrowleft \\ I(D) & & \text{Cat} \\ \curvearrowleft & \downarrow & \curvearrowright \\ & \mathcal{C}' & \end{array}$$

This yields the following definition. For simplicity, we assume that $\mathcal{C}_\emptyset = \mathcal{C}'_\emptyset$.

Definition. A *strict morphism* of D -categories $\mathcal{C}, \mathcal{C}'$ is the datum of

- for any $B \subseteq D$, a functor $H_B : \mathcal{C}_B \rightarrow \mathcal{C}'_B$
- for any $B \subseteq B'$, a natural transformation

$$\begin{array}{ccc} \mathcal{C}_{B'} & \xrightarrow{H_{B'}} & \mathcal{C}'_{B'} \\ F_{BB'} \downarrow & \nearrow \gamma_{BB'} & \downarrow F'_{BB'} \\ \mathcal{C}_B & \xrightarrow{H_B} & \mathcal{C}'_B \end{array} \quad (4.1)$$

such that

- $H_\emptyset = \text{id}$
- $\gamma_{BB} = \text{id}_{H_B}$
- For any $B \subseteq B' \subseteq B''$,

$$\gamma_{BB''} = \gamma_{BB'} \circ \gamma_{B'B''}$$

where \circ is the composition of natural transformations defined by

$$\begin{array}{ccc}
 \mathcal{C}_{B''} & \longrightarrow & \mathcal{C}'_{B''} \\
 \downarrow & \swarrow & \downarrow \\
 \mathcal{C}_{B'} & \longrightarrow & \mathcal{C}'_{B'} \\
 \downarrow & \swarrow & \downarrow \\
 \mathcal{C}_B & \longrightarrow & \mathcal{C}'_B
 \end{array} \quad (4.2)$$

The diagram (4.1), with $B = \emptyset$, induces an algebra homomorphism $E'_{B'} \rightarrow E_B$ which, by (4.2) is compatible with the maps $E_B \rightarrow E_{B'}$ and $E'_B \rightarrow E'_{B'}$ for any $B \subseteq B'$. As pointed out in [14, 3.3], this condition is too restrictive and will be weakened in the next paragraph.

4.4. Morphisms of D -categories.

Definition. A *morphism* of D -categories $\mathcal{C}, \mathcal{C}'$, with $\mathcal{C}_\emptyset = \mathcal{C}'_\emptyset$, is the datum of

- for any $B \subseteq D$ a functor $H_B : \mathcal{C}_B \rightarrow \mathcal{C}'_B$
- for any $B \subseteq B'$ and $\mathcal{F} \in \text{Mns}(B, B')$, a natural transformation

$$\begin{array}{ccc}
 \mathcal{C}_{B'} & \xrightarrow{H_{B'}} & \mathcal{C}'_{B'} \\
 F_{BB'} \downarrow & \gamma_{BB'}^{\mathcal{F}} \swarrow & \downarrow F'_{BB'} \\
 \mathcal{C}_B & \xrightarrow{H_B} & \mathcal{C}'_B
 \end{array}$$

such that

- $H_\emptyset = \text{id}$
- $\gamma_{BB}^{\mathcal{F}} = \text{id}_{H_B}$
- for any $B \subseteq B' \subseteq B''$, $\mathcal{F} \in \text{Mns}(B, B')$, $\mathcal{G} \in \text{Mns}(B', B'')$,

$$\gamma_{BB'}^{\mathcal{F}} \circ \gamma_{B'B''}^{\mathcal{G}} = \gamma_{BB''}^{\mathcal{F} \cup \mathcal{G}}$$

Remark. For any $\mathcal{F} \in \text{Mns}(B')$, the natural transformation $\gamma_{B'}^{\mathcal{F}}$ induces an algebra homomorphism $\Psi_{B'}^{\mathcal{F}} : E'_{B'} \rightarrow E_{B'}$ such that the following diagram commutes for any $B \in \mathcal{F}$

$$\begin{array}{ccc}
 E'_{B'} & \xrightarrow{\Psi_{B'}^{\mathcal{F}}} & E_{B'} \\
 \uparrow & & \uparrow \\
 E'_B & \xrightarrow{\Psi_B^{\mathcal{F}}} & E_B
 \end{array}$$

In particular, the collection of homomorphisms $\{\Psi_B^{\mathcal{F}}\}$ defines a morphism of D -algebras $E'_D \rightarrow E_D$ in the sense of [14, 3.4].

Remark. The above definition may be rephrased as follows. Let $M(D)$ be the category with objects the subdiagrams $B \subseteq D$ and morphisms $\text{Hom}(B', B) = \text{Mns}(B', B)$, with composition given by union. There is a forgetful functor $M(D) \rightarrow I(D)$ which is the identity on objects and maps $\mathcal{F} \in \text{Mns}(B', B)$ to the inclusion $B \subseteq B'$. Given two D -categories $\mathcal{C}, \mathcal{C}' : I(D) \rightarrow \text{Cat}$ a morphism $\mathcal{C} \rightarrow \mathcal{C}'$ as

defined above coincides with a morphism of the functors $M(D) \rightarrow \mathbf{Cat}$ given by the composition

$$M(D) \longrightarrow I(D) \xrightleftharpoons[c']{c} \mathbf{Cat}$$

4.5. Weak quasi-Coxeter categories.

Definition. A *weak quasi-Coxeter category of type D*

$$\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\}, \{\Upsilon_{\mathcal{F}\mathcal{G}}\})$$

is the datum of

- a D -category $\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\})$;
- for any pair $(\mathcal{F}, \mathcal{G})$ in $\mathbf{Mns}(B, B')$, a natural transformation (commonly referred to as *De Concini-Procesi associator*)

$$\Upsilon_{\mathcal{F}\mathcal{G}} \in \mathbf{Aut}(F_{BB'})$$

satisfying the following conditions

- **Orientation.** For any pair $(\mathcal{F}, \mathcal{G})$,

$$\Upsilon_{\mathcal{G}\mathcal{F}} = \Upsilon_{\mathcal{F}\mathcal{G}}^{-1}$$

- **Transitivity.** For any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbf{Mns}(B, B')$

$$\Upsilon_{\mathcal{F}\mathcal{G}} = \Upsilon_{\mathcal{F}\mathcal{H}} \Upsilon_{\mathcal{H}\mathcal{G}}$$

- **Factorization.** The assignment

$$\Upsilon : \mathbf{Mns}(B, B')^2 \rightarrow \mathbf{Aut}(F_{B'B})$$

is compatible with the embedding

$$\cup : \mathbf{Mns}(B, B') \times \mathbf{Mns}(B', B'') \rightarrow \mathbf{Mns}(B, B'')$$

for any $B'' \subset B' \subset B$, *i.e.*, the diagram

$$\begin{array}{ccc} \mathbf{Mns}(B, B')^2 \times \mathbf{Mns}(B', B'')^2 & \xrightarrow{\Upsilon \times \Upsilon} & \mathbf{Aut}(F_{B''B'}) \times \mathbf{Aut}(F_{B'B}) \\ \cup \downarrow & & \downarrow \circ \\ \mathbf{Mns}(B, B'')^2 & \xrightarrow{\Upsilon} & \mathbf{Aut}(F_{B''B}) \end{array}$$

is commutative.

Remark. To rephrase the above definition, consider the 2-category $\mathbf{qC}(D)$ obtained by adding to $M(D)$ a unique 2-isomorphism $\varphi_{\mathcal{G}\mathcal{F}}^{BB'} : \mathcal{F} \rightarrow \mathcal{G}$ for any pair of 1-morphisms $\mathcal{F}, \mathcal{G} \in \mathbf{Mns}(B', B)$, with the compositions

$$\varphi_{\mathcal{H}\mathcal{G}}^{BB'} \circ \varphi_{\mathcal{G}\mathcal{F}}^{BB'} = \varphi_{\mathcal{H}\mathcal{F}}^{BB'} \quad \text{and} \quad \varphi_{\mathcal{F}_2\mathcal{G}_2}^{BB'} \circ \varphi_{\mathcal{F}_1\mathcal{G}_1}^{B'B''} = \varphi_{\mathcal{F}_2 \cup \mathcal{F}_1, \mathcal{G}_2 \cup \mathcal{G}_1}^{BB''}$$

where $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathbf{Mns}(B', B)$, $B \subset B' \subseteq B''$ and $\mathcal{F}_1, \mathcal{G}_1 \in \mathbf{Mns}(B'', B')$, $\mathcal{F}_2, \mathcal{G}_2 \in \mathbf{Mns}(B', B)$. There is a unique functor $\mathbf{qC}(D) \rightarrow I(D)$ extending $M(D) \rightarrow I(D)$. A weak quasi-Coxeter category is the same as a 2-functor $\mathbf{qC}(D) \rightarrow \mathbf{Cat}$ fitting in a diagram

$$\begin{array}{ccc} \mathbf{qC}(D) & \xrightarrow{\quad} & \mathbf{Cat} \\ & \searrow & \nearrow \\ & I(D) & \end{array}$$

Note that, for any $B \subset B'$, the category $\text{Hom}_{\mathbf{qC}(D)}(B', B)$ is the 1-groupoid of the De Concini–Procesi associahedron on B'/B [14].

4.6. Quasi–Coxeter categories.

Definition. A *labelling* of the diagram D is the assignment of an integer $m_{ij} \in \{2, 3, \dots, \infty\}$ to any pair i, j of distinct vertices of D such that

$$m_{ij} = m_{ji} \quad m_{ij} = 2$$

if and only if $i \perp j$.

Let D be a labeled diagram.

Definition. A *quasi–Coxeter category of type D*

$$\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\}, \{\Upsilon_{\mathcal{FG}}\}, \{S_i\})$$

is the datum of

- a weak quasi–Coxeter category $\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\}, \{\Upsilon_{\mathcal{FG}}\})$;
- for any vertex $i \in \mathbf{V}(D)$, an element

$$S_i \in \text{Aut}(F_i)$$

satisfying the *braid relations*, *i.e.*, for any pairs i, j of distinct vertices of B , such that $2 < m_{ij} < \infty$, and $\mathcal{F}, \mathcal{G} \in \text{Mns}(B)$ such that $i \in \mathcal{F}, j \in \mathcal{G}$, the following relations hold in $\text{End}(F_B)$

$$\text{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}})(S_i) \cdot S_j \cdots = S_j \cdot \text{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}})(S_i) \cdots$$

where, by abuse of notation, we denote by S_i its image in $\text{End}(F_B)$ and the number of factors in each side equals m_{ij} .

The elements S_i will be commonly referred at as *local monodromies*.

Remark. It is clear that the factorization property implies the support and forgetful properties as stated in [14, Def. 3.12].

- **Support.** For any elementary pair $(\mathcal{F}, \mathcal{G})$ in $\text{Mns}(B, B')$, let $S = \text{supp}(\mathcal{F}, \mathcal{G})$, $Z = \mathfrak{z} \text{supp}(\mathcal{F}, \mathcal{G}) \subseteq D$ and

$$\tilde{\mathcal{F}} = \mathcal{F}|_{\mathfrak{z} \text{supp}(\mathcal{F}, \mathcal{G})}^{\text{supp}(\mathcal{F}, \mathcal{G})} \quad \tilde{\mathcal{G}} = \mathcal{G}|_{\mathfrak{z} \text{supp}(\mathcal{F}, \mathcal{G})}^{\text{supp}(\mathcal{F}, \mathcal{G})}$$

Then

$$\Upsilon_{\mathcal{FG}} = \text{id}_{BZ} \circ \Upsilon_{\tilde{\mathcal{F}}\tilde{\mathcal{G}}} \circ \text{id}_{B'S}$$

where the expression above denotes the composition of natural transformations

$$\begin{array}{ccc}
 \mathcal{C}_{B'} & & \mathcal{C}_{B'} \\
 \downarrow F_{BB'} & & \downarrow F_{SB'} \\
 \mathcal{C}_B & \xrightarrow{\Upsilon_{\mathcal{G}\mathcal{F}}} & \mathcal{C}_S \\
 \uparrow F_{BB'} & & \uparrow F_{ZS} \\
 \mathcal{C}_B & & \mathcal{C}_Z \\
 & & \downarrow F_{BZ} \\
 & & \mathcal{C}_B
 \end{array}$$

- **Forgetfulness.** For any equivalent elementary pairs $(\mathcal{F}, \mathcal{G}), (\mathcal{F}', \mathcal{G}')$ in $\text{Mns}(B, B')$

$$\Upsilon_{\mathcal{F}\mathcal{G}} = \Phi_{\mathcal{F}'\mathcal{G}'}$$

4.7. Morphisms of quasi-Coxeter categories.

Definition. A *morphism of quasi-Coxeter categories* $\mathcal{C}, \mathcal{C}'$ of type D is a morphism (H, γ) of the underlying D -categories such that

- For any $i \in B$, the corresponding morphism $\Psi_i : E'_i \rightarrow E_i$ satisfies

$$\Psi_i(S'_i) = S_i$$

- For any $\mathcal{F}, \mathcal{G} \in \text{Mns}(B, B')$,

$$H_B(\Upsilon_{\mathcal{F}\mathcal{G}}) \circ \gamma_{BB'}^{\mathcal{F}} \circ (\Upsilon'_{\mathcal{G}\mathcal{F}})_{H_{B'}} = \gamma_{BB'}^{\mathcal{G}}$$

in $\text{Nat}(F'_{BB'} \circ H_{B'}, H_B \circ F_{BB'})$, as in the diagram

$$\begin{array}{ccc}
 \mathcal{C}_{B'} & \xrightarrow{H_{B'}} & \mathcal{C}'_{B'} \\
 \downarrow \Upsilon_{\mathcal{F}\mathcal{G}} & \nearrow \gamma^{\mathcal{F}} & \downarrow \Upsilon'_{\mathcal{F}\mathcal{G}} \\
 \mathcal{C}_B & \xrightarrow{H_B} & \mathcal{C}'_B
 \end{array}$$

Remark. Note that the above condition can be alternatively stated in terms of morphisms Ψ_F as the identity

$$\Psi_{\mathcal{G}} \circ \text{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}}) = \text{Ad}(\Upsilon'_{\mathcal{G}\mathcal{F}}) \circ \Psi_{\mathcal{F}}$$

4.8. Strict monoidal D -categories.

Definition. A *strict monoidal D -category* $\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\}, \{J_{BB'}\})$ is a D -category $\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\})$ where

- for any $B \subseteq D$, $(\mathcal{C}_B, \otimes_B)$ is a strict monoidal category
- for any $B \subseteq B'$, the functor $F_{BB'}$ is endowed with a tensor structure $J_{BB'}$

with the additional condition that, for every $B \subseteq B' \subseteq B''$, $J_{BB'} \circ J_{B'B''} = J_{BB''}$.

Remark. The tensor structure J^B induces on $E_B = \text{End}(F_B)$ a *coproduct* $\Delta_B : E_B \rightarrow E_B^2$ where $E_B^2 = \text{End}(F_B^2)$ and $F_B^2 = \otimes \circ (F_B \times F_B)$, given by

$$\{g_V\}_{V \in \mathcal{C}_B} \mapsto \{\Delta_B(g)_{VW} = \text{Ad}(J_{VW}^B)(g_{V \otimes W})\}_{V, W \in \mathcal{C}_B}$$

Moreover, for any $B \subseteq B'$, E_B is a subbialgebra of $E_{B'}$, *i.e.*, the following diagram is commutative

$$\begin{array}{ccc} E_B & \xrightarrow{\Delta_B} & E_B^2 \\ \downarrow & & \downarrow \\ E_{B'} & \xrightarrow{\Delta_{B'}} & E_{B'}^2 \end{array}$$

Remark. Note that a strict monoidal D -category can be thought of as functor

$$\mathcal{C} : I(D) \rightarrow \text{Cat}_0^\otimes$$

preserving orthogonality, where Cat_0^\otimes denotes the 2-category of strict monoidal category, with monoidal functors and gauge transformations.

Definition. A morphism of strict monoidal D -categories is a natural transformation of the corresponding 2-functors $M(D) \rightarrow \text{Cat}_0^\otimes$, obtained by composition with $M(D) \rightarrow I(D)$.

4.9. Monoidal D -categories.

Definition. A *monoidal D -category*

$$\mathcal{C} = (\{(\mathcal{C}_B, \otimes_B, \Phi_B)\}, \{F_{BB'}\}, \{J_{BB'}^\mathcal{F}\})$$

is the datum of

- A D -category $(\{\mathcal{C}_B\}, \{F_{BB'}\})$ such that each $(\mathcal{C}_B, \otimes_B, \Phi_B)$ is a tensor category, with \mathcal{C}_\emptyset a strict tensor category, *i.e.*, $\Phi_\emptyset = \text{id}$.
- for any pair $B \subseteq B'$ and $\mathcal{F} \in \text{Mns}(B, B')$, a tensor structure $J_{BB'}^\mathcal{F}$ on the functor $F_{BB'} : \mathcal{C}_{B'} \rightarrow \mathcal{C}_B$

with the additional condition that, for any $B \subseteq B' \subseteq B''$, $\mathcal{F} \in \text{Mns}(B'', B')$, $\mathcal{G} \in \text{Mns}(B', B)$,

$$J_{BB'}^\mathcal{G} \circ J_{B'B''}^\mathcal{F} = J_{BB''}^{\mathcal{F} \cup \mathcal{G}}$$

Remark. The usual comparison with the algebra of endomorphisms leads to a collection of bialgebras $(E_B, \Delta_\mathcal{F}, \varepsilon)$ endowed with multiple coproducts, indexed by $\text{Mns}(B)$.

Remark. A monoidal D -category can be thought of as a functor $M(D) \rightarrow \text{Cat}^\otimes$ fitting in a diagram

$$\begin{array}{ccc} M(D) & \longrightarrow & \text{Cat}^\otimes \\ \downarrow & \swarrow & \downarrow \\ I(D) & \longrightarrow & \text{Cat} \end{array}$$

Accordingly, a morphism of monoidal D -categories is one of the corresponding functors.

$$\begin{array}{ccc} & \overset{c}{\curvearrowright} & \\ M(D) & \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} & \text{Cat}^\otimes \\ & \underset{c'}{\curvearrowleft} & \end{array}$$

4.10. Fibered monoidal D -categories. We shall often be concerned with monoidal D -categories such that the underlying categories $(\mathcal{C}_B, \otimes_B)$ are strict, and the functors $F_{BB'} : (\mathcal{C}_{B'}, \otimes_{B'}) \rightarrow (\mathcal{C}_B, \otimes_B)$ are tensor functors. This may be described in terms of the category $M(D)$ as follows. Let DCat^\otimes be the 2-category of Drinfeld categories, that is *strict* tensor categories (\mathcal{C}, \otimes) (in the sense of 2.6) endowed with an additional associativity constraint Φ making $(\mathcal{C}, \otimes, \Phi)$ a monoidal category. There is a canonical forgetful 2-functor $\text{DCat}^\otimes \rightarrow \text{Cat}^\otimes$.

We shall say that a monoidal D -category *fibers over* a strict monoidal D -category if the corresponding functor $M(D) \rightarrow \text{Cat}^\otimes$ maps into DCat^\otimes and fits in a commutative diagram

$$\begin{array}{ccc} M(D) & \longrightarrow & \text{DCat}^\otimes \\ \downarrow & \swarrow & \downarrow \\ I(D) & \longrightarrow & \text{Cat}^\otimes \end{array}$$

In this case, the coproduct $\Delta_{\mathcal{F}}$ on a bialgebra $\text{End}(F_B)$ is the twist of a reference coassociative coproduct Δ_0 on $E_D \text{End}(F_B)$ such that $\Delta_0 : E_B \rightarrow E_B^2 = \text{End}(F_B^2)$.

4.11. Braided monoidal D -categories.

Definition. A *braided monoidal D -category*

$$\mathcal{C} = (\{(\mathcal{C}_B, \otimes_B, \Phi_B, \beta_B)\}, \{(F_{BB'}, J_{BB'}^{\mathcal{F}})\})$$

is the datum of

- a monoidal D -category $(\{(\mathcal{C}_B, \otimes_B, \Phi_B)\}, \{(F_{BB'}, J_{BB'}^{\mathcal{F}})\})$
- for every $B \subseteq D$, a commutativity constraint β_B in \mathcal{C}_B , defining a braiding in $(\mathcal{C}_B, \otimes_B, \Phi_B)$.

Remark. Note that the tensor functors $(F_{BB'}, J_{BB'}^{\mathcal{F}}) : \mathcal{C}_{B'} \rightarrow \mathcal{C}_B$ are *not* assumed to map the commutativity constraint $\beta_{B'}$ to β_B .

Definition. A morphism of braided monoidal D -categories from \mathcal{C} to \mathcal{C}' is a morphism of the underlying monoidal D -categories such that the functors $H_B : \mathcal{C}_B \rightarrow \mathcal{C}'_B$ are braided tensor functors.

Remark. The fact that H_B are braided tensor functors automatically implies that

$$\Psi_{\mathcal{F}}^{\otimes 2}((R_B)_{J_{\mathcal{F}}}) = (R'_B)_{J'_{\mathcal{F}}}$$

in analogy with [14], where $R_B = (12) \circ \beta_B$. We assume that $\mathcal{C}_{\emptyset} = \mathcal{C}'_{\emptyset}$ is a symmetric strict tensor category.

4.12. Braided Quasi-Coxeter categories.

Definition. A *braided quasi-Coxeter category of type D*

$$\mathcal{C} = (\{(\mathcal{C}_B, \otimes_B, \Phi_B, \beta_B)\}, \{(F_{BB'}, J_{BB'}^{\mathcal{F}})\}, \{\Upsilon_{\mathcal{F}\mathcal{G}}\}, \{S_i\})$$

is the datum of

- a quasi-Coxeter category of type D ,

$$\mathcal{C} = (\{\mathcal{C}_B\}, \{F_{BB'}\}, \{\Upsilon_{\mathcal{F}\mathcal{G}}\}, \{S_i\})$$

- a braided monoidal D -category

$$\mathcal{C} = (\{(\mathcal{C}_B, \otimes_B, \Phi_B, \beta_B)\}, \{(F_{BB'}, J_{BB'}^{\mathcal{F}})\})$$

satisfying the following conditions

- for any $B \subseteq B'$, and $\mathcal{G}, \mathcal{F} \in \text{Mns}(B, B')$, the natural transformation $\Upsilon_{\mathcal{F}\mathcal{G}} \in \text{Aut}(F_{BB'})$ determines an isomorphism of tensor functors $(F_{BB'}, J_{BB'}^{\mathcal{G}}) \rightarrow (F_{BB'}, J_{BB'}^{\mathcal{F}})$, that is for any $V, W \in \mathcal{C}_{B'}$,

$$(\Upsilon_{\mathcal{G}\mathcal{F}})_{V \otimes W} \circ (J_{BB'}^{\mathcal{F}})_{V, W} = (J_{BB'}^{\mathcal{G}})_{V, W} \circ ((\Upsilon_{\mathcal{G}\mathcal{F}})_V \otimes (\Upsilon_{\mathcal{G}\mathcal{F}})_W)$$

- for any $i \in D$, the following holds in $\text{Aut}(F_i \otimes F_i)$:

$$J_i^{-1} \cdot F_i(\beta_i) \cdot \Delta_i(S_i) \cdot J_i = \beta_{\emptyset} \cdot (S_i \otimes S_i)$$

A morphism of braided quasi-Coxeter categories of type D is a morphism of the underlying quasi-Coxeter categories and braided monoidal D -categories.

Remark. A braided quasi-Coxeter category of type D determines a 2-functor $\text{qC}(D) \rightarrow \text{Cat}^{\otimes}$ fitting in a diagram

$$\begin{array}{ccc} \text{qC}(D) & \longrightarrow & \text{Cat}^{\otimes} \\ \downarrow & \swarrow & \downarrow \\ I(D) & \longrightarrow & \text{Cat} \end{array}$$

Note however that this functor does not entirely capture the braided quasi-Coxeter structure since it does not encode the commutativity constraints β_B and automorphisms S_i .

4.13. Representations of generalised braid groups. Let D be a labeled diagram.

Definition. The *Artin-Tits braid group B_D* is the group generated by elements S_i labeled by the vertices $i \in D$ with relations

$$\underbrace{S_i S_j \cdots}_{m_{ij}} = \underbrace{S_j S_i \cdots}_{m_{ij}}$$

for any $i \neq j$ such that $m_{ij} < \infty$. We shall also refer to B_D as the braid group corresponding to D .

The axiomatic of a braided quasi-Coxeter category \mathcal{C} encodes the joint action of the braid groups B_D and B_n , $n \geq 0$, on the objects in \mathcal{C} . More precisely,

Proposition. *Let \mathcal{C} be a braided, quasi-Coxeter category of type D . There is a collection of homomorphisms $\lambda_{\mathcal{F}} : B_D \rightarrow \text{Aut}(F_D)$, labelled by maximal nested sets of D , which is uniquely determined by*

$$\lambda_{\mathcal{F}}(S_i) = S_i \quad \text{if} \quad \{i\} \in \mathcal{F}$$

and

$$\lambda_{\mathcal{G}} = \text{Ad}(\Upsilon_{\mathcal{GF}}) \circ \lambda_{\mathcal{F}}$$

A similar statement describing the action of the braid groups B_n holds for any $B \subset D$ and any $n \geq 0$.

5. QUANTISED KAC-MOODY ALGEBRAS

We show that for every symmetrisable Kac-Moody algebra, the category of representations of the quantum group provides an example of braided quasi-Coxeter categories, encoding the action of the universal R -matrix and the quantum Weyl group operators.

5.1. Kac-Moody algebras. Denote by \mathbf{k} a field of characteristic zero. We recall definitions from [11] and [9]. Let $A = (a_{ij})_{i,j \in \mathbf{I}}$ be an $n \times n$ symmetrisable matrix with entries in \mathbf{k} , i.e. there exists a (fixed) collection of nonzero numbers $\{d_i\}_{i \in \mathbf{I}}$ such that $d_i a_{ij} = d_j a_{ji}$ for all $i, j \in \mathbf{I}$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A . It means that \mathfrak{h} is a vector space of dimension $2n - \text{rank}(A)$, $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{h_1, \dots, h_n\} \subset \mathfrak{h}$ are linearly independent, and $(\alpha_i, h_j) = a_{ji}$.

Definition. The Lie algebra $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(A)$ is generated by $\mathfrak{h}, \{e_i, f_i\}_{i \in \mathbf{I}}$ with defining relations

$$\begin{aligned} [h, h'] &= 0 \quad h, h' \in \mathfrak{h}; & [h, e_i] &= (\alpha_i, h)e_i \\ [h, f_i] &= -(\alpha_i, h)f_i; & [e_i, f_j] &= \delta_{ij}h_i \end{aligned}$$

There exists a unique maximal ideal \mathfrak{r} in $\tilde{\mathfrak{g}}$ that intersect \mathfrak{h} trivially. Let $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{r}$. The algebra \mathfrak{g} is called *generalised Kac-Moody algebra*. The Lie algebra \mathfrak{g} is graded by principal gradation $\deg(e_i) = 1, \deg(f_i) = -1, \deg(\mathfrak{h}) = 0$, and the homogenous component are all finite-dimensional.

Let us now choose a non-degenerate bilinear symmetric form on \mathfrak{h} such that $\langle h, h_i \rangle = d_i^{-1}(\alpha_i, h)$. There exists a unique extension of the form \langle, \rangle to an invariant symmetric bilinear form on $\tilde{\mathfrak{g}}$. For this extension, one gets $\langle e_i, f_j \rangle = \delta_{ij}d_i^{-1}$. The kernel of this form on $\tilde{\mathfrak{g}}$ is \mathfrak{r} , therefore it descends to a non-degenerate bilinear form on \mathfrak{g} .

5.2. Diagrammatic subalgebras. Let \mathbf{J} be a nonempty subset of \mathbf{I} . Consider the submatrix of A defined by

$$A_{\mathbf{J}} = (a_{ij})_{i,j \in \mathbf{J}}$$

We recall the following proposition from [11, Ex.1.2]

Proposition. *Let*

$$\Pi_{\mathbf{J}} = \{\alpha_j \mid j \in \mathbf{J}\} \quad \Pi_{\mathbf{J}}^{\vee} = \{h_j \mid j \in \mathbf{J}\}$$

Let $\mathfrak{h}'_{\mathbf{J}}$ be the subspace of \mathfrak{h} generated by $\Pi_{\mathbf{J}}^{\vee}$ and

$$\mathfrak{t}_{\mathbf{J}} = \bigcap_{j \in \mathbf{J}} \text{Ker } \alpha_j = \{h \in \mathfrak{h} \mid \langle \alpha_j, h \rangle = 0 \ \forall j \in \mathbf{J}\}$$

Let $\mathfrak{h}''_{\mathbf{J}}$ be a supplementary subspace of $\mathfrak{h}'_{\mathbf{J}} + \mathfrak{t}_{\mathbf{J}}$ in \mathfrak{h} and let

$$\mathfrak{h}_{\mathbf{J}} = \mathfrak{h}'_{\mathbf{J}} \oplus \mathfrak{h}''_{\mathbf{J}}$$

Then,

- (i) $(\mathfrak{h}_{\mathbf{J}}, \Pi_{\mathbf{J}}, \Pi_{\mathbf{J}}^{\vee})$ *is a realization of the generalised Cartan matrix $A_{\mathbf{J}}$.*
- (ii) *The subalgebra $\mathfrak{g}_{\mathbf{J}} \subset \mathfrak{g}$, generated by $\{e_j, f_j\}_{j \in \mathbf{J}}$ and $\mathfrak{h}_{\mathbf{J}}$, is the Kac-Moody algebra associated to the realization $(\mathfrak{h}_{\mathbf{J}}, \Pi_{\mathbf{J}}, \Pi_{\mathbf{J}}^{\vee})$ of $A_{\mathbf{J}}$.*

Set

$$Q_{\mathbf{J}} = \sum_{j \in \mathbf{J}} \mathbb{Z} \alpha_j \subset Q \quad \mathfrak{g} = \mathfrak{g}(A) = \bigoplus_{\alpha \in Q} \mathfrak{g}_{\alpha}$$

Then,

(iii)

$$\mathfrak{g}_{\mathbf{J}} = \mathfrak{h}_{\mathbf{J}} \oplus \bigoplus_{\alpha \in Q_{\mathbf{J}} \setminus \{0\}} \mathfrak{g}_{\alpha}$$

Let A be a symmetrisable matrix with a fixed decomposition and $(-|-)$ be the standard normalised non-degenerate bilinear form on \mathfrak{h} . Then,

- (iv) *The restriction of $(-|-)$ to $\mathfrak{h}_{\mathbf{J}}$ is non-degenerate.*

Proof. Since $\dim(\mathfrak{h}'_{\mathbf{J}} \cap \mathfrak{t}_{\mathbf{J}}) = \dim(\mathfrak{z}(\mathfrak{g}_{\mathbf{J}})) = n_{\mathbf{J}} - l_{\mathbf{J}}$, where $n_{\mathbf{J}} = |\mathbf{J}|$ and $l_{\mathbf{J}} = \text{rank}(A_{\mathbf{J}})$, it follows that

$$\dim \mathfrak{h}''_{\mathbf{J}} = n_{\mathbf{J}} - l_{\mathbf{J}} \quad \dim \mathfrak{h}_{\mathbf{J}} = 2n_{\mathbf{J}} - l_{\mathbf{J}}$$

Moreover, by construction, the restriction of $\{\alpha_j\}_{j \in \mathbf{J}}$ to $\mathfrak{h}_{\mathbf{J}}$ are linearly independent. Indeed, since $\langle \sum c_j \alpha_j, \mathfrak{t}_{\mathbf{J}} \rangle = 0$ for all $c_j \in \mathbb{C}$,

$$\langle \sum_{j \in \mathbf{J}} c_j \alpha_j, \mathfrak{h}_{\mathbf{J}} \rangle = 0 \implies \langle \sum_{j \in \mathbf{J}} c_j \alpha_j, \mathfrak{h} \rangle = 0 \implies c_j = 0$$

This proves (i). The proof of (ii) and (iii) is clear.

Assume now that A is irreducible and symmetrisable and there exists $h \in \mathfrak{h}_{\mathbf{J}}$ such that

$$(h|h') = 0 \quad \forall h' \in \mathfrak{h}_{\mathbf{J}}$$

In particular, $(h|\alpha_j^{\vee}) = 0$ and $h \in \mathfrak{h}'_{\mathbf{J}} \cap \mathfrak{t}_{\mathbf{J}} \subset \mathfrak{h}'_{\mathbf{J}}$. Therefore, $h = \sum c_j \alpha_j^{\vee}$ and

$$(\sum c_j \alpha_j^{\vee} | h') = \sum c_j (\alpha_j^{\vee} | h') = \langle \sum c_j d_j \alpha_j, h' \rangle = 0$$

Since the operators $\{\alpha_j\}$ are linearly independent over $\mathfrak{h}_{\mathbf{J}}$ and $d_j \neq 0$, we have $c_j = 0$ and $h = 0$. We conclude that (i) is non-degenerate on $\mathfrak{h}_{\mathbf{J}}$ and (iv) is proved. \square

5.3. D -algebra structures on Kac–Moody algebras. Let A be an irreducible, generalised Cartan matrix. Let $D = D(A)$ be the Coxeter–Dynkin diagram of \mathfrak{g} , that is, the connected graph having \mathbf{I} as vertex set and an edge between i and j if $a_{ij} \neq 0$. For any $i \in \mathbf{I}$, let $\mathfrak{sl}_2^i \subset \mathfrak{g}$ be the three-dimensional subalgebra spanned by e_i, f_i, h_i .

We want to study the existence of D -algebra structures on \mathfrak{g} , *i.e.*, we need a collection of subalgebras $\{\mathfrak{g}_B\}_{B \subset D}$ labelled by subdiagrams of D satisfying

$$[\mathfrak{g}_B, \mathfrak{g}_{B'}] = 0 \quad \text{if } B \perp B' \quad \text{and} \quad \mathfrak{g}_B \subset \mathfrak{g}_{B'} \quad \text{if } B \subset B'$$

Any subdiagram $B \subseteq D$ defines a subset $\mathbf{J}_B \subset \mathbf{I}$. We would like to use the assignment $\mathbf{J} \mapsto \mathfrak{g}_{\mathbf{J}}$ to define a D -algebra structure on $\mathfrak{g} = \mathfrak{g}(A)$.

Proposition. *Let \mathfrak{g} be a Kac–Moody algebra.*

- (i) *The derived algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is endowed with a standard D -structure.*
- (ii) *If \mathfrak{g} is of finite, affine type, there is a standard D -structure on \mathfrak{g} .*
- (iii) *If \mathfrak{g} is of hyperbolic type, there exists a D -structure on \mathfrak{g} .*
- (iv) *In full generality, it is not always possible to endow a Kac–Moody algebra with a D -structure.*

Proof. The main issue lies in the choice of the subspaces $\mathfrak{h}_{\mathbf{J}}''$. The derived algebra $\mathfrak{g}'_{\mathbf{J}} = [\mathfrak{g}_{\mathbf{J}}, \mathfrak{g}_{\mathbf{J}}]$ is generated by $\{e_j, f_j, h_j\}_{j \in \mathbf{J}}$, where $h_j = [e_j, f_j]$. Therefore, it does not depend of the choice of the subspace $\mathfrak{h}_{\mathbf{J}}''$. The assignment $\mathbf{J} \mapsto \mathfrak{g}'_{\mathbf{J}}$ defines a D -structure that coincides with the one provided in [14, 3.2.2].

For any subset \mathbf{J} of finite type, $\dim \mathfrak{h}_{\mathbf{J}}'' = n_{\mathbf{J}} - l_{\mathbf{J}} = 0$ and $\mathfrak{h}_{\mathbf{J}} = \mathfrak{h}'_{\mathbf{J}}$. Therefore, if A is a generalised Cartan matrix of finite type, $\mathfrak{h}_{\mathbf{J}}'' = \{0\}$ for any subset $\mathbf{J} \subset \mathbf{I}$. The D -algebra structure on $\mathfrak{g} = \mathfrak{g}(A)$ is then uniquely defined by the subalgebras $\{\mathfrak{sl}_2^i\}_{i \in \mathbf{I}}$ and the Cartan subalgebra is defined for any subdiagram $B \subset D$ by

$$\mathfrak{h}_B = \{h_i \mid i \in V(D)\}$$

If A is a generalised Cartan matrix of affine type, we obtain diagrammatic Cartan subalgebras \mathfrak{h}_B , where

$$\mathfrak{h}_B = \begin{cases} \{h_i \mid i \in V(B)\} & \text{if } B \subset D \\ \mathfrak{h} & \text{if } B = D \end{cases}$$

If A is an irreducible generalised Cartan matrix of hyperbolic type, *i.e.*, every submatrix is of finite or affine type, it is still possible to define a $D_{\mathfrak{g}}$ -algebra structure, depending upon the choice of the subspaces $\mathfrak{h}_{\mathbf{J}}''$ for $|\mathbf{I} \setminus \mathbf{J}| = 1$.

On the other hand, this is not always possible for an arbitrary generalised Cartan matrix of order ≥ 3 . It is easy to show that the symmetric irreducible Cartan matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

does not admit any D -algebra structure on $\mathfrak{g}(A)$, since $\dim \mathfrak{h}_{23} = 3$ and $\dim \mathfrak{h}_{123} \cap \mathfrak{h}_{234} = 2$.

In order to obtain a D -algebra structure on $\mathfrak{g} = \mathfrak{g}(A)$, we have to satisfy the following condition:

$$\mathfrak{h}_{\mathbf{J}} \subset \mathfrak{t}_{\mathbf{J}^\perp} \cap \bigcap_{\mathbf{J}' \subset \mathbf{J}} \mathfrak{h}_{\mathbf{J}'}$$

Since $\mathfrak{t}_{\mathbf{J}^\perp} + \mathfrak{t}_{\mathbf{J}} = \mathfrak{h}$, we can always choose $\mathfrak{h}_{\mathbf{J}} \subseteq \mathfrak{t}_{\mathbf{J}^\perp}$.

Lemma. *Assume given a $D_{\mathfrak{g}}$ -algebra structure on $\mathfrak{g} = \mathfrak{g}(A)$. Then for any two subsets $\mathbf{J}', \mathbf{J}'' \subset \mathbf{I}$,*

$$\text{corank}(\mathbf{A}_{\mathbf{J}' \cap \mathbf{J}''}) \leq \text{corank}(\mathbf{A}_{\mathbf{J}'} + \text{corank}(\mathbf{A}_{\mathbf{J}''})$$

In particular, if $\text{corank}(\mathbf{A}_{\mathbf{J}'} + \text{corank}(\mathbf{A}_{\mathbf{J}''}) = 0$, then $\text{corank}(\mathbf{A}_{\mathbf{J}' \cap \mathbf{J}''}) = 0$.

Proof. The result is an immediate consequence of the estimate, given by the construction,

$$\dim(\mathfrak{h}_{\mathbf{J}'} \cap \mathfrak{h}_{\mathbf{J}''}) \leq |\mathbf{J}' \cap \mathbf{J}''| + (\text{corank}(\mathbf{A}_{\mathbf{J}'} + \text{corank}(\mathbf{A}_{\mathbf{J}''}))$$

and the constraint

$$\mathfrak{h}_{\mathbf{J}' \cap \mathbf{J}''} \subseteq \mathfrak{h}_{\mathbf{J}'} \cap \mathfrak{h}_{\mathbf{J}''}$$

□

Unfortunately, the previous condition on the corank is not enough to obtain a D -algebra structure on $\mathfrak{g}(A)$. Consider the symmetric Cartan matrix

$$A = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

The matrix A clearly satisfies the above condition. Nonetheless, a suitable \mathfrak{h}_{12}'' , complement in \mathfrak{h} of $(\mathfrak{h}_{12}' + \mathfrak{t}_{12})$, should satisfies:

$$\mathfrak{h}_{12}'' \subset \mathfrak{h}_{123} = \mathfrak{h}_{12}' \quad \text{and} \quad \mathfrak{h}_{12}'' \subseteq \mathfrak{t}_4 = \langle \mathfrak{h}_{12}', -2\alpha_3^\vee + \alpha_4^\vee \rangle$$

which are not compatible conditions. Therefore, there is no suitable structure for A . □

5.4. Extended Kac–Moody algebras. Following a suggestion of P. Etingof, we give a modified definition of \mathfrak{g} , along the lines of [10], characterized by a bigger Cartan subalgebra.

Let $A = (a_{ij})_{i,j \in I} \in M_n(\mathbb{Z})$ be a generalised Cartan matrix.

Definition. The *extended Kac–Moody algebra* of A is the \mathbf{k} -algebra $\overline{\mathfrak{g}} = \overline{\mathfrak{g}}(A)$ with generators $e_i, f_i, h_i, \lambda_i^\vee$, $i \in I$, and defining relations

- $[h_i, h_j] = [\lambda_i^\vee, \lambda_j^\vee] = [h_i, \lambda_j^\vee] = 0$
- $[h_i, e_j] = a_{ij}e_j, [h_i, f_j] = -a_{ji}f_j$
- $[\lambda_i^\vee, e_j] = \delta_{ij}e_j, [\lambda_i^\vee, f_j] = -\delta_{ij}f_j$
- $\text{ad}(x_i)^{1-a_{ij}}(x_j) = 0$ for $i \neq j$ and $x = e, f$

As in the Kac–Moody case, $\overline{\mathfrak{g}}$ is naturally endowed with a standard bilinear form.

Proposition. *Let A be a symmetrisable generalised Cartan matrix of rank l , $D = D(A)$ its Dynkin diagram, \mathfrak{g} the corresponding Kac–Moody algebra. Let $\{d_r\}_{r=l+1}^n$ be the completion of $\{h_i\}_{i=1}^n$ to a basis of $\mathfrak{h} \subset \mathfrak{g}$ defined by the relations*

$$[d_r, d_s] = 0 = [d_r, h_i] \quad [d_r, e_i] = \delta_{ir}e_i \quad [d_r, f_i] = -\delta_{ir}f_i$$

- (i) $\bar{\mathfrak{g}} = \bar{\mathfrak{g}}_D$ has a canonical structure of D-algebra, given by the collection of subalgebras $\{\bar{\mathfrak{g}}_B\}_{B \subset D}$, where

$$\bar{\mathfrak{g}}_B = \langle e_i, f_i, h_i, \lambda_i^\vee \mid i \in B \rangle$$

- (ii) There is a canonical embedding $\mathfrak{g} \subset \bar{\mathfrak{g}}$ mapping

$$e_i, f_i, h_i, d_r \mapsto e_i, f_i, h_i, \lambda_r^\vee$$

$i = 1, \dots, n, r = l, \dots, n$. The embedding preserves the bilinear forms on $\mathfrak{g}, \bar{\mathfrak{g}}$.

- (iii) $\mathfrak{g}' = \bar{\mathfrak{g}}'$.

- (iv) The triple $(\bar{\mathfrak{g}} \oplus \bar{\mathfrak{h}}, \bar{\mathfrak{b}}_\pm)$ is a graded Manin triple, with bilinear form $(,) = (,)_{\bar{\mathfrak{g}}} - (,)_{\bar{\mathfrak{h}}}$.

- (v) $\bar{\mathfrak{g}}$ has a standard Lie bialgebra structure with cobracket

$$\delta|_{\bar{\mathfrak{h}}} = 0 \quad \delta(e_i) = \frac{d_i}{2} e_i \wedge h_i \quad \delta(f_i) = \frac{d_i}{2} f_i \wedge h_i$$

From now on, we avoid the notation $\bar{\mathfrak{g}}$, and we denote an extended Kac–Moody algebra by \mathfrak{g} .

5.5. Drinfeld–Yetter modules. Let \mathfrak{b} be the Borel subalgebra of \mathfrak{g} generated by \mathfrak{h} and the elements $e_i, i \in \mathbf{I}$. As a Lie bialgebra, \mathfrak{b} has a natural category of representations, called Drinfeld–Yetter modules and denoted $\mathrm{DY}_{\mathfrak{b}}$. A triple (V, π, π^*) is a Drinfeld–Yetter \mathfrak{b} -module if (V, π) is a module and (V, π^*) is a comodule, satisfying the compatibility condition

$$\pi^* \circ \pi - \mathrm{id} \otimes \pi \circ (12) \circ \mathrm{id} \otimes \pi^* = [,]_{\mathfrak{b}} \otimes \mathrm{id} \circ \mathrm{id} \otimes \pi^* - \mathrm{id} \otimes \pi \circ \delta_{\mathfrak{b}} \otimes \mathrm{id}$$

$\mathrm{DY}_{\mathfrak{b}}$ has an obvious structure of symmetric tensor categories, and it can be identified with a subcategory of modules over the Drinfeld double of \mathfrak{b} . The classical r -matrix of the latter acts on a tensor product $V \otimes W$ of Drinfeld–Yetter module as the map $r_{VW} : V \otimes W \rightarrow V \otimes W$ defined by

$$r_{VW} = (\pi_V \otimes \mathrm{id}) \circ (12) \circ (\mathrm{id} \otimes \pi_W^*)$$

Following [5], one can define a structure of braided monoidal category on $\mathrm{DY}_{\mathfrak{b}}$ depending on the choice of a Lie associator Φ . The commutativity constraint is explicitly defined by the formula

$$\beta_{VW} = (12) \circ e^{\frac{\hbar}{2} \Omega_{VW}} \quad \Omega_{VW} = r_{VW} + \sigma_{12} \circ r_{WV} \circ \sigma_{12}$$

We denote this braided tensor category $\mathrm{DY}_{\mathfrak{b}}^{\Phi}$, where $\Phi_{\mathfrak{b}} = \Phi(\Omega_{\mathfrak{b}})$.

5.6. Quantisation of extended Kac–Moody algebras. Let \mathfrak{g} be a complex symmetrisable extended Kac–Moody algebra with generalised Cartan matrix $A = (a_{ij})_{i,j \in \mathbf{I}}$. The Drinfeld–Jimbo quantum group of \mathfrak{g} is the quantised universal enveloping algebra $U_{\hbar} \mathfrak{g}$ topologically generated over $\mathbb{C}[[\hbar]]$ by \mathfrak{h} and elements $E_i, F_i, i \in \mathbf{I}$ with relations

$$[h, h'] = 0 \quad [h, E_i] = \alpha_i(h) E_i \quad [h, F_i] = -\alpha_i(h) F_i \quad [E_i, F_i] = \frac{q_i^{h_i} - q_i^{-h_i}}{q_i - q_i^{-1}}$$

for any $h, h' \in \mathfrak{h}$, and

$$\sum_{m=0}^{1-a_{ij}} \frac{(-1)^m}{[m]_{q_i}! [1-a_{ij}-m]_{q_i}!} E_i^{1-a_{ij}-m} E_j E_i^m = 0$$

(analogously for F_i 's). The coproduct is defined by

$$\Delta|_{\mathfrak{h}} = 0 \quad \Delta(E_i) = E_i \otimes q_i^{h_i} + 1 \otimes E_i \quad \Delta(F_i) = F_i \otimes 1 + q_i^{-h_i} \otimes F_i$$

It is well-known that the subalgebra $U_{\mathfrak{h}}\mathfrak{b}$, generated by \mathfrak{h} and $E_i, i \in \mathbf{I}$, is endowed with a symmetric non-degenerate invariant bilinear form with values in $\mathbb{C}((\hbar))$, and $U_{\mathfrak{h}}\mathfrak{g}$ is isomorphic to a quotient of the restricted quantum double of $U_{\mathfrak{h}}\mathfrak{b}$.

There is a notion of Drinfeld–Yetter module on a Hopf algebra. A Drinfeld–Yetter $U_{\mathfrak{h}}\mathfrak{b}$ -module is a triple $(\mathcal{V}, \pi, \pi^*)$ such that (\mathcal{V}, π) is a module and (\mathcal{V}, π^*) is a comodule, satisfying the compatibility condition

$$\pi^* \circ \pi = (m^{(3)} \otimes \pi) \circ \sigma_{(13)(24)} \circ (S^{-1} \otimes \text{id}^{\otimes 4}) \circ (\Delta^{(3)} \otimes \pi^*)$$

The category of Drinfeld–Yetter $U_{\mathfrak{h}}\mathfrak{b}$ -modules $\text{DY}_{U_{\mathfrak{h}}\mathfrak{b}}$ has a natural structure of braided tensor category with braiding $\beta = \sigma \circ R$, where

$$R = (\pi \otimes \text{id}) \circ \sigma_{12} \circ (\text{id} \otimes \pi^*)$$

$\text{DY}_{U_{\mathfrak{h}}\mathfrak{b}}$ is a subcategory of modules over the quantum double of $U_{\mathfrak{h}}\mathfrak{b}$. Under this identification the morphism R corresponds to the action of the universal R -matrix $DU_{\mathfrak{h}}\mathfrak{b}$.

5.7. Quasi-Coxeter structures on quantum groups. Let $A = (a_{ij})_{i,j \in \mathbf{I}} \in M_n(\mathbb{Z})$ be a generalised Cartan matrix, $D = D(A)$ the Coxeter–Dynkin diagram, $\mathfrak{g} = \mathfrak{g}(A)$ the extended Kac–Moody algebra over \mathbb{C} , \mathfrak{b} the positive Borel, and $U_{\mathfrak{h}}\mathfrak{g}, U_{\mathfrak{h}}\mathfrak{b}$ the corresponding Drinfeld–Jimbo quantum groups.

Proposition. *The category $\text{DY}_{\mathfrak{h}}^{\text{int}}$ of \mathfrak{h} -semisimple integrable Drinfeld–Yetter $U_{\mathfrak{h}}\mathfrak{b}$ -modules has a natural structure of braided quasi-Coxeter category.*

5.7.1. Drinfeld–Yetter modules. Let $\text{DY}_{U_{\mathfrak{h}}\mathfrak{b}}$ be the category of Drinfeld–Yetter $U_{\mathfrak{h}}\mathfrak{b}$ -modules. We consider the full subcategory $\text{DY}_{U_{\mathfrak{h}}\mathfrak{b}}^{\text{int}}$, whose objects satisfy the following conditions:

- (i) For any $\mathcal{V} \in \text{DY}_{U_{\mathfrak{h}}\mathfrak{b}}^{\text{int}}$, there is a decomposition as \mathfrak{h} -modules

$$\mathcal{V} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathcal{V}_{\lambda}$$

where $\mathcal{V}_{\lambda} = \{v \in \mathcal{V} \mid h \cdot v = \lambda(h)v\}$

- (ii) The action of the elements E_i, F_i 's on \mathcal{V} is locally finite.

5.7.2. D-structure. For any $B \subset B' \subseteq D$, there are inclusions of Hopf algebras $U_{\mathfrak{h}}\mathfrak{g}_B \subset U_{\mathfrak{h}}\mathfrak{g}_{B'} \subset U_{\mathfrak{h}}\mathfrak{g}$. These induce a restriction functor

$$\text{Res}_{BB'} : \text{DY}_{U_{\mathfrak{h}}\mathfrak{b}_{B'}}^{\text{int}} \rightarrow \text{DY}_{U_{\mathfrak{h}}\mathfrak{b}_B}^{\text{int}}$$

satisfying the obvious relation for any $B \subset B' \subset B''$

$$\text{Res}_{BB'} \circ \text{Res}_{B'B''} = \text{Res}_{BB''}$$

5.7.3. Braided monoidal structure. For any $B \subseteq D$, the universal R -matrix $R_B^{\mathfrak{h}}$ of the quasitriangular Hopf algebra $U_{\mathfrak{h}}\mathfrak{g}_B$ endows the category $\text{DY}_{U_{\mathfrak{h}}\mathfrak{b}_B}^{\text{int}}$ a braided monoidal structure with commutativity constraint $\beta_B = \sigma \circ R_B^{\mathfrak{h}}$ and trivial associativity constraint $\Phi_B = \text{id}^{\otimes 3}$. It follows that the restriction functors $\text{Res}_{BB'}$ have a trivial tensor structure with $J_{BB'}^{\mathcal{F}} = \text{id}^{\otimes 2}$.

5.7.4. *Quasi-Coxeter structure.* By definition, the quantum Weyl group operator of $U_{\hbar}\mathfrak{g}$ corresponding to $i \in \mathbf{I}$ is the operator $S_i^{\hbar} \in \text{End}(\text{Res}_{\emptyset i})$ acting on $\mathcal{V} \in \text{DY}_{U_{\hbar}\mathfrak{b}}^{\text{int}}$ as

$$S_i^{\hbar}(v) = \sum_{\substack{a,b,c \in \mathbb{Z} \\ a-b+c = -\lambda(\alpha_i^{\vee})}} (-1)^b q_i^{\frac{\hbar^2}{4} + b - ac} E_i^{(a)} F_i^{(b)} E_i^{(c)} v$$

where

$$E_i^{(a)} = \frac{E_i^a}{[a]_i!} \quad F_i^{(a)} = \frac{F_i^a}{[a]_i!}$$

and $v \in \mathcal{V}_{\lambda}$ for $\lambda \in \mathfrak{h}^*$ [12]. By [12], the quantum Weyl group operators satisfy the braid relations

$$\underbrace{S_i^{\hbar} S_j^{\hbar} \cdots}_{m_{ij}} = \underbrace{S_j^{\hbar} S_i^{\hbar} \cdots}_{m_{ij}}$$

and the coproduct identity

$$\Delta^{21}(S_i^{\hbar}) = R_i \cdot (S_i^{\hbar} \otimes S_i^{\hbar})$$

It follows that the quantum Weyl group operators of $U_{\hbar}\mathfrak{b}$ define a quasi-Coxeter structure on $\text{DY}_{U_{\hbar}\mathfrak{b}}$ with trivial associators $\Upsilon_{\mathcal{F}\mathcal{G}} = \text{id}$.

6. AN EQUIVALENCE OF QUASI-COXETER CATEGORIES

6.1. In this section we prove the main result of the paper. We transfer the quasi-Coxeter structure of $U_{\hbar}\mathfrak{g}$ on $U_{\hbar}\mathfrak{g}[[\hbar]]$. More precisely, we define an equivalence of braided quasi-Coxeter categories between the representation theories of $U_{\hbar}\mathfrak{g}$ and $U_{\hbar}\mathfrak{g}[[\hbar]]$.

Theorem. *Let \mathfrak{g} be an extended Kac-Moody algebra with positive Borel \mathfrak{b} , and $U_{\hbar}\mathfrak{b}$ the corresponding Drinfeld-Jimbo quantum group. For any choice of a Lie associator Φ , there exists a braided quasi-Coxeter structure on the category DY^{int} of \mathfrak{h} -semisimple integrable Drinfeld-Yetter \mathfrak{b} -modules and an equivalence of braided quasi-Coxeter categories*

$$\text{DY}^{\text{int}} \xrightarrow{\simeq} \text{DY}_{\hbar}^{\text{int}}$$

The structures on $\text{DY}_{\hbar}^{\text{int}}$ and DY^{int} have the following description:

$$\text{DY}_{\hbar}^{\text{int}} = (\{(\text{DY}_{U_{\hbar}\mathfrak{b}_B}^{\text{int}}, \otimes_B, \text{id}, R_B^{\hbar})\}, \{(\text{Res}_{BB'}, \text{id})\}, \{\text{id}\}, \{S_i^{\hbar}\})$$

where R_B^{\hbar} is the universal R -matrix and S_i^{\hbar} are the quantum Weyl group operators, and

$$\text{DY}^{\text{int}} = (\{(\text{DY}_{\mathfrak{b}_B}^{\text{int}}, \otimes_B, \Phi_B, R_B)\}, \{(\text{Res}_{BB'}, J_{BB'}^{\mathcal{F}})\}, \{\Upsilon_{\mathcal{G}\mathcal{F}}\}, \{S_{i,C}\})$$

where ⁷

$$\begin{aligned} S_{i,C} &= \tilde{s}_i \exp\left(\frac{\hbar}{2} \cdot C_i\right) \\ R_B &= \exp\left(\frac{\hbar}{2} \Omega_B\right) \\ \text{Alt}_2 J_{BB'}^{\mathcal{F}} &= \frac{\hbar}{2} \left(\frac{r_{B'} - r_{B'}^{21}}{2} - \frac{r_B - r_B^{21}}{2} \right) \end{aligned}$$

and $\Upsilon_{\mathcal{GF}}$, $J_{BB'}^{\mathcal{F}}$ are weight zero elements.

The proof is carried out in 6.2–6.8.

6.2. Braided D-structure on DY^{int} . Let D be the Coxeter–Dynkin diagram of \mathfrak{g} and Φ a fixed Lie associator. The braided quasi–Coxeter structure of the quantum group is the same as in 5.7. On the classical side, one consider, for any $B \subseteq D$, the category DY_B of Drinfeld–Yetter \mathfrak{b}_B -modules. For any $B \subset B'$, there is a restriction functor

$$\text{Res}_{BB'} : \text{DY}_{B'} \rightarrow \text{DY}_B$$

which preserves integrability and weight space decomposition.

It is easy to see that, if $B, B' \subseteq D$ and $B \perp B'$,

$$\text{DY}_{B \cup B'} = \text{OFP}(\text{Vect}, \text{DY}_B, \text{DY}_{B'})$$

in the sense of Section 2.

The braided D-structure is obtained by considering the Drinfeld category $\mathcal{C}_B = \text{DY}_B^{\Phi}$ of Drinfeld–Yetter \mathfrak{b}_B -modules with associator $\Phi_{\mathfrak{b}_B} = \Phi(\Omega_B)$ and R -matrix $R_B = \exp(\hbar/2\Omega_B)$.

6.3. Relative twists. In [1, Thm. 3.1], we construct a tensor structure $J_{BB'}$ on $\text{Res}_{BB'}$ which satisfies

$$\text{Alt}_2 J_{BB'} = \frac{\hbar}{2} \left(\frac{r_{B'} - r_{B'}^{21}}{2} - \frac{r_B - r_B^{21}}{2} \right)$$

The composition of the relative twists along a maximal chain

$$\mathbf{C} : B = C_0 \subset C_1 \subset \cdots \subset C_r = B'$$

with $|C_k \setminus C_{k-1}| = 1$ defines on $\text{Res}_{BB'}$ a tensor structure $J_{BB'}^{\mathbf{C}}$.

We explained in Section 3 that a maximal chain defines uniquely a maximal nested set $\mathcal{F}_{\mathbf{C}} \in \text{Mns}(B, B')$, but this is not a one to one correspondence. For example, for $D = A_3$, the maximal nested set

$$\mathcal{F} = \{\{\alpha_1\}, \{\alpha_3\}, \{\alpha_1, \alpha_2, \alpha_3\}\}$$

corresponds to two different chains of maximal length

$$\mathbf{C}_1 : \{\alpha_1\} \subset \{\alpha_1\} \sqcup \{\alpha_3\} \subset A_3 \quad \mathbf{C}_2 : \{\alpha_3\} \subset \{\alpha_1\} \sqcup \{\alpha_3\} \subset A_3$$

⁷ In the completion $\widehat{Usl_2^h[[\hbar]]}$ with respect to DY_i^{int} , one sets

$$\tilde{s}_i = \exp(e_i) \exp(-f_i) \exp(e_i) \quad C_i = \frac{(\alpha_i, \alpha_i)}{2} (e_i f_i + f_i e_i + \frac{1}{2} h_i^2)$$

In particular, we have to show that the twists $J_{BB'}^{\mathbf{C}}$ depend only on the maximal nested set corresponding to \mathbf{C} . Equivalently, we have to show that for any $B_1 \perp B_2$, the construction of the fiber functor

$$\begin{array}{ccc}
 & \mathcal{C}_{B_1 \sqcup B_2} & \\
 F_{B_2, B_1 \sqcup B_2} \swarrow & \downarrow & \searrow F_{B_1, B_1 \sqcup B_2} \\
 \mathcal{C}_{B_1} & & \mathcal{C}_{B_2} \\
 F_{B_1} \searrow & \downarrow & \swarrow F_{B_2} \\
 & \mathcal{C}_\emptyset &
 \end{array}$$

is independent of the choice of the chain. In our case,

$$\mathcal{C}_{B_1 \sqcup B_2} = \text{OFP}(\text{Vect}, \text{DY}_{B_1}, \text{DY}_{B_2})$$

and the braided tensor structure is given by product of the braided tensor structures on

$$\mathcal{C}_{B_1} = \text{DY}_{\mathfrak{b}_{B_1}}^{\Phi_{B_1}} \quad \mathcal{C}_{B_2} = \text{DY}_{\mathfrak{b}_{B_2}}^{\Phi_{B_2}}$$

Similarly, the tensor structure on the restriction functor

$$\mathcal{C}_{B_1 \sqcup B_2} \rightarrow \mathcal{C}_{B_i} \quad i = 1, 2$$

is obtained killing the tensor structure on \mathcal{C}_{B_j} , $j = 2, 1$, *i.e.*, applying the tensor structure on $\mathcal{C}_{B_j} \rightarrow \mathcal{C}_\emptyset$. In particular, the tensor structure on $F_{B_1} \circ F_{B_1, B_1 \sqcup B_2}$ and $F_{B_2} \circ F_{B_2, B_1 \sqcup B_2}$ coincide, since $[\mathfrak{b}_{B_1}, \mathfrak{b}_{B_2}] = 0$.

Therefore, for any $B \subset B'$, we obtain a collection of twists $J_{BB'}^{\mathcal{F}}$, $\mathcal{F} \in \text{Mns}(B, B')$, which satisfies

$$J_{BB'}^{\mathcal{G}} \circ J_{B'B''}^{\mathcal{F}} = J_{BB''}^{\mathcal{G} \cup \mathcal{F}}$$

This shows that the collection

$$\left(\{\text{DY}_B^{\text{int}}, \otimes, \Phi_B, R_B\}, \{\text{Res}_{BB'}, J_{BB'}^{\mathcal{F}}\} \right)$$

defines a braided D-structure on DY^{int} .

6.4. Equivalence of braided D-categories. We now show that there exists an equivalence of braided D-categories from

$$\text{DY}^{\text{int}} = \left(\{\text{DY}_{\mathfrak{b}_B}^{\text{int}}, \otimes, \Phi_B, R_B\}, \{\text{Res}_{BB'}, J_{BB'}^{\mathcal{F}}\} \right)$$

to

$$\text{DY}_h^{\text{int}} = \left(\{\text{DY}_{U_h \mathfrak{b}_B}^{\text{int}}, \otimes, \text{id}^{\otimes 3}, R_B^h\}, \{\text{Res}_{BB'}^h, \text{id}^{\otimes 2}\} \right)$$

In [1, Thm. 4.6], for any Lie associator Φ and split embedding of Lie bialgebras $\mathfrak{a} \hookrightarrow \mathfrak{b} \rightarrow \mathfrak{a}$, we construct a natural transformation

$$\begin{array}{ccc}
 \text{DY}_{\mathfrak{b}}^{\Phi} & \xrightarrow{\tilde{F}_{\mathfrak{b}}} & \text{DY}_{U_h \mathfrak{b}} \\
 \text{Res}_{\mathfrak{b}, \mathfrak{a}} \downarrow & \nearrow \gamma & \downarrow \text{Res}_{\mathfrak{b}, \mathfrak{a}}^h \\
 \text{DY}_{\mathfrak{a}}^{\Phi} & \xrightarrow{\tilde{F}_{\mathfrak{a}}} & \text{DY}_{U_h \mathfrak{a}}
 \end{array}$$

where $\tilde{F}_{\mathfrak{b}}, \tilde{F}_{\mathfrak{a}}$ are the Etingof–Kazhdan equivalences of braided tensor categories for \mathfrak{a} and \mathfrak{b} , and the functor $\text{Res}_{\mathfrak{b}, \mathfrak{a}}$ is endowed with the tensor structure from [1, Thm.

3.1]. Applied to our case, this construction gives rise to an equivalence of braided D-categories 4.4.

This is the results of two pieces of data: a collection of braided tensor equivalences, indexed by subdiagrams of B , and a collection of natural transformation, indexed by maximal nested sets.

For any subdiagram $B \subseteq D$, one choose the Etingof–Kazhdan equivalence

$$\tilde{F}_B : \mathrm{DY}_B^\Phi \rightarrow \mathrm{DY}_{U_{\mathfrak{h}} \mathfrak{b}_B}$$

For any $B \subset B' \subseteq D$ and any maximal chain

$$\mathbf{C} : B = B_0 \subset B_1 \subset \cdots \subset B_r = B'$$

one gets a sequence of split embeddings

$$\mathfrak{b}_0 \subset \mathfrak{b}_1 \subset \cdots \subset \mathfrak{b}_r$$

and the natural transformations

$$\gamma_{k,k+1} \in \mathrm{Nat}_\otimes \left(\mathrm{Res}_{k,k+1}^{\tilde{F}_k} \circ \tilde{F}_{k+1}, \tilde{F}_k \circ \mathrm{Res}_{k,k+1} \right)$$

for $k = 0, \dots, r-1$, yield a natural transformation ⁸

$$\gamma_{\mathbf{C}} = \gamma_{0,1} \circ \cdots \circ \gamma_{r-1,r}$$

in $\mathrm{Nat}_\otimes(\mathrm{Res}_{0,1}^{\tilde{F}_0} \circ \cdots \circ \mathrm{Res}_{r-1,r}^{\tilde{F}_r} \circ \tilde{F}_r, \tilde{F}_0 \circ \mathrm{Res}_{0,1} \circ \cdots \circ \mathrm{Res}_{r-1,r})$. As before, we have to show that the natural transformation $\gamma_{\mathbf{C}}$ depends uniquely on the maximal nested sets $\mathcal{F}_{\mathbf{C}}$. For any $B_1 \sqcup B_2$, one consider the diagram

$$\begin{array}{ccc} & \mathcal{C}_{B_1 \sqcup B_2} & \\ F_{B_2, B_1 \sqcup B_2} \swarrow & \downarrow & \searrow F_{B_1, B_1 \sqcup B_2} \\ \mathcal{C}_{B_1} & & \mathcal{C}_{B_2} \\ F_{B_1} \searrow & \downarrow & \swarrow F_{B_2} \\ & \mathcal{C}_\emptyset & \end{array}$$

and it follows that, as in the case of the relative twists,

$$\gamma_{B_1} \circ \gamma_{B_1, B_1 \sqcup B_2} = \gamma_{B_2} \circ \gamma_{B_2, B_1 \sqcup B_2}$$

Therefore, for any maximal nested set $\mathcal{F} \in \mathrm{Mns}(B, B')$, there is a well defined a natural transformation

$$\gamma_{BB'}^{\mathcal{F}} \in \mathrm{Nat}_\otimes(\mathrm{Res}_{BB'}^{\tilde{F}_B} \circ \tilde{F}_{B'}, \tilde{F}_B \circ \mathrm{Res}_{BB'})$$

and the data $(\{\tilde{F}_B\}, \{\gamma_{BB'}^{\mathcal{F}}\})$ define an isomorphism of braided D-categories $\mathrm{DY}_{\mathfrak{b}_B}^{\Phi_B} \rightarrow \mathrm{DY}_{U_{\mathfrak{h}} \mathfrak{b}_B}$ which preserves \mathfrak{h} -action and integrability and restricts to

$$\mathrm{DY}^{\mathrm{int}} \rightarrow \mathrm{DY}_{\mathfrak{h}}^{\mathrm{int}}$$

⁸ In case $B = \emptyset$, $\mathrm{Res}_{0,1} : \mathrm{DY}_{\mathfrak{b}_1}^\Phi \rightarrow \mathrm{Vect}$ is the Etingof–Kazhdan fiber functor and $\tilde{F}_0 = \mathrm{id}$.

6.5. De Concini–Procesi associators. The equivalence of braided D–categories

$$(\tilde{F}_B, \gamma_{BB'}^{\mathcal{F}}) : \mathrm{DY}^{\mathrm{int}} \rightarrow \mathrm{DY}_h^{\mathrm{int}}$$

induces on

$$\mathrm{DY}^{\mathrm{int}} = (\{(\mathrm{DY}_B^{\mathrm{int}}, \otimes_B, \Phi_B, R_B)\}, \{(\mathrm{Res}_{BB'}, J_{BB'}^{\mathcal{F}})\})$$

a structure of quasi–Coxeter category of type D.

The De Concini–Procesi associators $\Upsilon_{\mathcal{G}\mathcal{F}}$ are easily obtained from the natural transformations $\gamma_{BB'}^{\mathcal{F}}$. For any $B \subset B'$ and $\mathcal{F}, \mathcal{G} \in \mathrm{Mns}(B, B')$, one defines $\Upsilon_{\mathcal{F}\mathcal{G}} \in \mathrm{End}(\mathrm{Res}_{BB'})$ by

$$\tilde{F}_B(\Upsilon_{\mathcal{F}\mathcal{G}}) = (\gamma_{BB'}^{\mathcal{F}})^{-1} \circ \gamma_{BB'}^{\mathcal{G}}$$

It is clear by construction that these satisfy the required properties:

(i) **Orientation.** For any $\mathcal{F}, \mathcal{G} \in \mathrm{Mns}(B, B')$

$$\Upsilon_{\mathcal{F}\mathcal{G}} = \Upsilon_{\mathcal{G}\mathcal{F}}^{-1}$$

(ii) **Transitivity.** For any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathrm{Mns}(B, B')$

$$\Upsilon_{\mathcal{F}\mathcal{G}} \circ \Upsilon_{\mathcal{G}\mathcal{H}} = \Upsilon_{\mathcal{F}\mathcal{H}}$$

(iii) **Factorization.** For any $\mathcal{F}, \mathcal{G} \in \mathrm{Mns}(B, B')$, $\mathcal{F}', \mathcal{G}' \in \mathrm{Mns}(B', B'')$,

$$\Upsilon_{(\mathcal{F} \cup \mathcal{F}')(\mathcal{G} \cup \mathcal{G}')} = \Upsilon_{\mathcal{F}\mathcal{G}} \circ \Upsilon_{\mathcal{F}'\mathcal{G}'}$$

and they satisfy Finally, the elements $\Upsilon_{\mathcal{G}\mathcal{F}}$ satisfy

$$\Delta(\Upsilon_{\mathcal{G}\mathcal{F}}) \circ J_{\mathcal{F}} = J_{\mathcal{G}} \circ \Upsilon_{\mathcal{G}\mathcal{F}}^{\otimes 2}$$

because they are given by composition of invertible natural tensor transformations.

6.6. Local monodromies. The braided quasi–Coxeter structure on $\mathrm{DY}^{\mathrm{int}}$ is completed by transferring the quantum Weyl group operators $S_i^h, i \in \mathbf{I}$. It follows from the previous construction that for any maximal nested set $\mathcal{F} \in \mathrm{Mns}(B)$ there exists an isomorphism of topological Hopf algebras

$$\Psi_{\mathcal{F}} : \mathrm{End}(\mathbf{f}_{B,h}) \rightarrow \mathrm{End}(\mathbf{f}_B)$$

where $\mathbf{f}_B : \mathrm{DY}_B^{\Phi} \rightarrow \mathrm{Vect}$, $\mathbf{f}_{h,B} : \mathrm{DY}_{U_h \mathbf{b}_B} \rightarrow \mathrm{Vect}$ are the usual fiber functors. One has $S_i^h \in \mathrm{End}(\mathbf{f}_{h,i})$ and one sets

$$S_i = \Psi_i(S_i^h)$$

We have to prove the compatibility relations of the elements $\Upsilon_{\mathcal{G}\mathcal{F}}, S_i$ with the underlying structure of braided D–category on $\mathrm{DY}^{\mathrm{int}}$.

Let $\mathcal{F} \in \mathrm{Mns}(\mathrm{D})$ such that $\{i\} \in \mathcal{F}$. The isomorphism $\Psi_{\mathcal{F}}$ is obtained by an equivalence of braided D–categories and therefore

$$\Psi_{\mathcal{F}}((R_i^h)_{\mathcal{F}}) = (R_i)_{\mathcal{F}}$$

Thus, the element S_i ’s satisfy the relation

$$\Delta_{\mathcal{F}}(S_i) = (R_i)_{\mathcal{F}}^{21} \cdot (S_i \otimes S_i)$$

The braid relations follows, since

$$\mathrm{Ad}(\Upsilon_{\mathcal{G}\mathcal{F}})\Psi_{\mathcal{F}} = \Psi_{\mathcal{G}}$$

6.7. Extension to Levi subalgebras. In analogy with [14, Thm. 9.1], we want to show that the relative twists and the Casimir associators are weight zero elements. This is equivalent to show that the corresponding tensor functors and the natural transformations lift to the level of Levi subalgebras:

$$\mathfrak{g}_B \subset \mathfrak{l}_B = \mathfrak{n}_{B,+} \oplus \mathfrak{h} \oplus \mathfrak{n}_{B,-} \subset \mathfrak{g}$$

This is easily obtained once one proves that the Etingof–Kazhdan functor preserves abelian subbialgebras.

6.7.1. Abelian Lie bialgebras and central extensions. We now consider the following special case, that generalises the role of Levi subalgebras for Kac–Moody algebras.

Proposition. *Let $\mathfrak{a} \hookrightarrow \mathfrak{l} \hookrightarrow \mathfrak{b}$ be a split inclusion of Lie bialgebras, where \mathfrak{l} is a central extension of \mathfrak{a} . Then the relative twists and the gauge transformations from [1, Thm. 3.1 - 4.6] are invariant under \mathfrak{l} . In particular, the Etingof–Kazhdan functor $\tilde{F}_{\mathfrak{b}}$ preserves abelian subbialgebras.*

Proof. We use the notation from [1]. For $\mathfrak{a} = \{0\}$, the statement reduces to prove that the Etingof–Kazhdan functor preserves the action of an abelian subbialgebra $\mathfrak{l} \subset \mathfrak{b}$ (cf. [9, Thm. 4.3], with $\mathfrak{l} = \mathfrak{h}$). Under this assumption, the natural map $U\mathfrak{l} \rightarrow U_{\hbar}\mathfrak{b}$ defines an inclusion of bialgebras. For any $V \in \mathrm{DY}_{\mathfrak{b}}^{\Phi}$, the natural identification

$$\alpha_V : \tilde{F}_{\mathfrak{b}}(V) \rightarrow V$$

is then an isomorphism of Drinfeld–Yetter \mathfrak{l} -modules. This gives the following commutative diagram

$$\begin{array}{ccc} \mathrm{DY}_{\mathfrak{b}}^{\Phi} & \xrightarrow{\tilde{F}_{\mathfrak{b}}} & \mathrm{DY}_{U_{\hbar}\mathfrak{b}} \\ & \searrow & \swarrow \\ & \mathrm{DY}_{\mathfrak{l}} & \\ & \downarrow & \\ & \mathrm{Vect} & \end{array}$$

$F_{\mathfrak{b}}$ (curved arrow from $\mathrm{DY}_{\mathfrak{b}}^{\Phi}$ to Vect)

and shows that the twists and the natural transformations are \mathfrak{l} -invariant.

In the general case, the tensor restriction functor fits in an analogous diagram. One observes that the restriction functor $\Gamma : \mathrm{DY}_{\mathfrak{b}}^{\Phi} \rightarrow \mathrm{DY}_{\mathfrak{a}}^{\Phi}$ factors as follows:

$$\begin{array}{ccc} \mathrm{DY}_{\mathfrak{b}}^{\Phi} & \xrightarrow{\quad} & \mathrm{DY}_{\Gamma(L_-)} \\ & \searrow \Gamma & \swarrow \\ & \mathrm{DY}_{\mathfrak{a}}^{\Phi} & \end{array}$$

where $\Gamma(L_-)$ is the braided Hopf algebra in $\mathrm{DY}_{\mathfrak{a}}^{\Phi}$ constructed in [1, Sec. 5]. The category $\mathrm{DY}_{\Gamma(L_-)}$ is naturally equivalent to the category of Drinfeld–Yetter module over the Radford product $\Gamma(L_-) \otimes U\mathfrak{a}_-$ with an inclusion of bialgebras

$$U\mathfrak{l} \subset \Gamma(L_-) \otimes U\mathfrak{a}$$

One gets a commutative diagram

$$\begin{array}{ccc}
 \mathrm{DY}_b^\Phi & \xrightarrow{\quad} & \mathrm{DY}_{\Gamma(L_-)} \\
 & \searrow \quad \swarrow & \\
 & \mathrm{DY}_l^\Phi & \\
 & \downarrow & \\
 & \mathrm{DY}_a^\Phi &
 \end{array}
 \quad \begin{array}{c} \Gamma \\ \curvearrowright \end{array}$$

This proves that relative twists are invariant under l . It is clear that the Casimir operator $\Omega_a \in (\mathfrak{g}_a \otimes \mathfrak{g}_a)^{\mathfrak{gl}}$ defines a braided tensor structure on DY_l that is preserved by the restriction functor induced by the inclusion $j : \mathfrak{a} \subset l$. Given the decomposition $l = \mathfrak{a} \rtimes \mathfrak{c}$, the natural map $U\mathfrak{c} \rightarrow \mathrm{End}_a(j^*V)$ induces an action of $U\mathfrak{c}$ on $\tilde{F}_a(j^*V)$, commuting with the action of $U_h\mathfrak{a}$. Therefore, we obtain a commutative diagram

$$\begin{array}{ccc}
 \mathrm{DY}_l^\Phi & \xrightarrow{\tilde{F}_a} & \mathrm{DY}_{U_h l} \\
 j_B^* \downarrow & & \downarrow \\
 \mathrm{DY}_a^\Phi & \xrightarrow{\tilde{F}_a} & \mathrm{DY}_{U_h a}
 \end{array}$$

where \tilde{F}_a is the tensor functor induced by the composition $\tilde{F}_a \circ j^*$. The natural transformation γ automatically lifts to the level of l , as showed in the following diagram

$$\begin{array}{ccccc}
 \mathrm{DY}_b^\Phi & & \xrightarrow{\tilde{F}} & & \mathrm{DY}_{U_h b} \\
 & \searrow & & \swarrow & \\
 & & \mathrm{DY}_l^\Phi & \xrightarrow{\tilde{F}_a} & \mathrm{DY}_{U_h l} \\
 & \swarrow j^* & & \searrow & \\
 \mathrm{DY}_a^\Phi & & \xrightarrow{\tilde{F}_a} & & \mathrm{DY}_{U_h a}
 \end{array}
 \quad \begin{array}{c} \Gamma \\ \downarrow \end{array}$$

□

6.7.2. *Weight zero property.* Finally, we get

Proposition. *The relative twists and the Casimir associators are weight zero elements.*

Proof. For $B = \emptyset$, the statement reduces to prove that the Etingof–Kazhdan functor preserves the \mathfrak{h} -action [9, Thm. 4.3]. For $B \neq \emptyset$, the result is a consequence of Proposition 6.7.1 applied to Levi subalgebras. □

6.8. **Normalised isomorphisms.** In the completion $\widehat{U\mathfrak{sl}_2^i[[\hbar]]}$ with respect to $\mathrm{DY}_i^{\mathrm{int}}$, there are preferred element $S_{i,C}$

$$S_{i,C} = \tilde{s}_i \exp\left(\frac{\hbar}{2} C_i\right)$$

where

$$\tilde{s}_i = \exp(e_i) \exp(-f_i) \exp(e_i) \quad C_i = \frac{(\alpha_i, \alpha_i)}{2} (e_i f_i + f_i e_i + \frac{1}{2} h_i^2)$$

It remains to show that there exists an equivalence of quasi-Coxeter categories between

$$\mathrm{DY}^{\mathrm{int}} = (\{(\mathrm{DY}_B^{\mathrm{int}}, \otimes_B, \Phi_B, R_B)\}, \{(\mathrm{Res}_{BB'}, J_{BB'}^{\mathcal{F}})\}, \{\Upsilon_{\mathcal{GF}}\}, \{S_{i,C}\})$$

and

$$\mathrm{DY}_h^{\mathrm{int}} = (\{(\mathrm{DY}_{U_h \mathfrak{b}_B}^{\mathrm{int}}, \otimes_B, \mathrm{id}, R_B^h)\}, \{(\mathrm{Res}_{BB'}^h, \mathrm{id})\}, \{\mathrm{id}\}, \{S_i^h\})$$

It is enough to prove that the natural transformation γ_i

$$\begin{array}{ccc} \mathrm{DY}_i^{\mathrm{int}} & \xrightarrow{\tilde{F}_i} & \mathrm{DY}_{i,h}^{\mathrm{int}} \\ & \searrow f & \swarrow \gamma_i \\ & \mathcal{A} & \swarrow f_h \end{array}$$

can be modified in such a way that the induced isomorphism at the level of endomorphism algebras $\widehat{U_h \mathfrak{sl}_2^i} \rightarrow \widehat{U \mathfrak{sl}_2^i}[[\hbar]]$ maps S_i^h to $S_{i,C}$. The Etingof–Kazhdan isomorphism

$$\Psi_i^{\mathrm{EK}} : \widehat{U_h \mathfrak{sl}_2^i} \rightarrow \widehat{U \mathfrak{sl}_2^i}[[\hbar]]$$

is the identity mod \hbar and the identity on the Cartan subalgebra. Set $S_i = \Psi_i^{\mathrm{EK}}(S_i^h)$. Then $S_i \equiv \tilde{s}_i \pmod{\hbar}$ and, by [14, Proposition 8.1, Lemma 8.4], we have

$$S_i^2 = S_{i,C}^2 \quad S_i = \mathrm{Ad}(x)(S_{i,C})$$

for $x = (S_{i,C} \cdot S_i^{-1})^{\frac{1}{2}}$. The modified isomorphism

$$\Psi_i = \mathrm{Ad}(x) \circ \Psi_i^{\mathrm{EK}}$$

maps S_i^h to $S_{i,C}$. Moreover, Ψ_i correspond with the natural transformation given by the composition of γ_i with $x \in \widehat{U \mathfrak{sl}_2^i}[[\hbar]] = \mathrm{End}(f)$

$$\begin{array}{ccccc} \mathrm{DY}_i^{\mathrm{int}} & \xlongequal{\quad} & \mathrm{DY}_i^{\mathrm{int}} & \xrightarrow{\tilde{F}} & \mathrm{DY}_{i,h}^{\mathrm{int}} \\ \downarrow f & \xleftarrow{x} & \downarrow f & \xleftarrow{\gamma_i} & \downarrow f_h \\ \mathrm{Vect} & \xlongequal{\quad} & \mathrm{Vect} & \xlongequal{\quad} & \mathrm{Vect} \end{array}$$

The result follows substituting γ_i with $x \circ \gamma_i$.

6.9. Main theorem. We summarize the proof outlined in the previous paragraphs.

Theorem. *Let \mathfrak{g} be an extended Kac–Moody algebra with positive Borel \mathfrak{b} , and $U_h \mathfrak{b}$ the corresponding Drinfeld–Jimbo quantum group. For any choice of a Lie associator Φ , there exists a braided quasi-Coxeter structure on the category $\mathrm{DY}^{\mathrm{int}}$ of \mathfrak{h} -semisimple integrable Drinfeld–Yetter \mathfrak{b} -modules and an equivalence of braided quasi-Coxeter categories*

$$\mathrm{DY}^{\mathrm{int}} \xrightarrow{\simeq} \mathrm{DY}_h^{\mathrm{int}}$$

The structures on $\mathrm{DY}_h^{\mathrm{int}}$ and $\mathrm{DY}^{\mathrm{int}}$ have the following description:

$$\mathrm{DY}_h^{\mathrm{int}} = (\{(\mathrm{DY}_{U_h \mathfrak{b}_B}^{\mathrm{int}}, \otimes_B, \mathrm{id}, R_B^h)\}, \{(\mathrm{Res}_{BB'}^h, \mathrm{id})\}, \{\mathrm{id}\}, \{S_i^h\})$$

where R_B^{\hbar} is the universal R -matrix and S_i^{\hbar} are the quantum Weyl group operators, and

$$\mathrm{DY}^{\mathrm{int}} = (\{(\mathrm{DY}_{\mathfrak{b}_B}^{\mathrm{int}}, \otimes_B, \Phi_B, R_B)\}, \{(\mathrm{Res}_{BB'}, J_{BB'}^{\mathcal{F}})\}, \{\Upsilon_{\mathcal{GF}}\}, \{S_{i,C}\})$$

where

$$\begin{aligned} S_{i,C} &= \tilde{s}_i \exp\left(\frac{\hbar}{2} \cdot C_i\right) \\ R_B &= \exp\left(\frac{\hbar}{2} \Omega_B\right) \\ \mathrm{Alt}_2 J_{BB'}^{\mathcal{F}} &= \frac{\hbar}{2} \left(\frac{r_{B'} - r_{B'}^{21}}{2} - \frac{r_B - r_B^{21}}{2} \right) \end{aligned}$$

and $\Upsilon_{\mathcal{GF}}, J_{BB'}^{\mathcal{F}}$ are weight zero elements.

Proof. The braided quasi-Coxeter structure on $\mathrm{DY}_{\hbar}^{\mathrm{int}}$ is defined in 5.7. The braided D-structure on $\mathrm{DY}^{\mathrm{int}}$ is defined in 6.2 and 6.3. The property of the R -matrices R_B follows by construction; the property of the relative twists follows by [1, Sec. 3]. The equivalence of braided D-categories between $\mathrm{DY}_{\hbar}^{\mathrm{int}}$ and $\mathrm{DY}^{\mathrm{int}}$ is constructed in 6.4. This is used to transfer the remaining data of the braided quasi-Coxeter structure of $\mathrm{DY}_{\hbar}^{\mathrm{int}}$ on $\mathrm{DY}^{\mathrm{int}}$. The De Concini-Procesi associators are discussed in 6.5; the local monodromies in 6.6. The weight zero property of the relative twists and the Casimir associators is proved in Proposition 6.7.1, 6.7.2. Finally, the local monodromies are normalised in 6.8. This complete the proof of Theorem 6.1. \square

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